

THE DEPTH OF THE MAXIMAL SUBGROUPS OF REE GROUPS

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To the memory of Professor K. Corrádi

ABSTRACT. We determine the combinatorial and the ordinary depth of the maximal subgroups of the simple Ree groups, $R(q)$.

Key words: Ree groups, combinatorial depth, ordinary depth, maximal subgroup.

AMS MSC2000 classification: 20D06, 20C15, 20B20, 20C33

1. INTRODUCTION

Similarly to [3], [4], [5] and [9], we will study the combinatorial and ordinary depth of some subgroups of a certain group class. We will determine the depth of maximal subgroups of Ree groups $R(q)$, $q \geq 27$.

Originally depth was defined for von-Neumann algebras, see [6]. Later it was also defined for Hopf algebras, see [16]. For some recent results in this direction, see [13] and [14]. In [15] and later in [2] the depth of semisimple algebra inclusions was studied. The ordinary depth of a group inclusion $H \leq G$ (denoted by $d(H, G)$) is defined as the depth of group algebra inclusion $\mathbb{C}H \subseteq \mathbb{C}G$, studied in [1]. It is proved in [1] that the ordinary depth of a subgroup H of a finite group G is bounded from above by the combinatorial depth $d_c(H, G)$. In particular, it is always finite. For the original definitions of $d_c(H, G)$ and of $d(H, G)$, see [1]. Here we will use some equivalent forms of the original definitions.

Let us denote by $H^x := x^{-1}Hx$ and $H^{(x_1, \dots, x_n)} := H \cap H^{x_1} \cap \dots \cap H^{x_n}$, for $x_1, \dots, x_n \in G$. Let us denote by $\mathcal{U}_0(H) := H$, $\mathcal{U}_1(H) := \{H \cap H^x | x \in G\}$, and in general let $\mathcal{U}_n(H) := \{H^{(x_1, \dots, x_n)} | x_1, \dots, x_n \in G\}$

Definition 1.1. [1, Thm. 3.9] *Let H be a subgroup of finite group G . The combinatorial depth $d_c(H, G)$ is the minimal possible positive integer which can be determined from the following upper bounds:*

- (1) *Let $i \geq 1$. The combinatorial depth $d_c(H, G) \leq 2i$ if and only if for any $x_1, \dots, x_i \in G$, there exist $y_1, \dots, y_{i-1} \in G$ with $H^{(x_1, \dots, x_i)} = H^{(y_1, \dots, y_{i-1})}$. (In other words, $d_c(H, G) \leq 2i$ if and only if $\mathcal{U}_i(H) = \mathcal{U}_{i-1}(H)$)*
- (2) *Let $i > 1$. The combinatorial depth $d_c(H, G) \leq 2i - 1$ if and only if for any $x_1, \dots, x_i \in G$, there exist $y_1, \dots, y_{i-1} \in G$ with $H^{(x_1, \dots, x_i)} = H^{(y_1, \dots, y_{i-1})}$ and $x_1 h x_1^{-1} = y_1 h y_1^{-1}$ for any $h \in H^{(x_1, \dots, x_i)}$.*
- (3) *The combinatorial depth $d_c(H, G) = 1$ if and only if for every $x \in G$ there exists an element $y \in H$ such that $x h x^{-1} = y h y^{-1}$, for every $h \in H$. (In other words: $G = HC_G(H)$.)*

It is easy to see that $d_c(H, G) \leq 2$ if and only if $H \triangleleft G$.

We define the ordinary depth using an equivalent form of it, more details are in [2].

Two irreducible characters $\alpha, \beta \in \text{Irr}(H)$ are *related*, $\alpha \sim \beta$, if they are constituents of $\chi|_H$, for some $\chi \in \text{Irr}(G)$. The *distance* $d(\alpha, \beta) = m$ is the smallest integer m such that there is a chain of irreducible characters of H such that $\alpha = \psi_0 \sim \psi_1 \sim \dots \sim \psi_m = \beta$. If there is no such chain then $d(\alpha, \beta) = -\infty$ and if $\alpha = \beta$ then the distance is zero. If X is the set of irreducible constituents of $\chi|_H$ then $m(\chi) := \max_{\alpha \in \text{Irr}(H)} \min_{\psi \in X} d(\alpha, \psi)$.

Definition 1.2. [2, Thm 3.9, Thm 3.13][15] *Let H be a subgroup of a finite group G . The ordinary depth $d(H, G)$ is the minimal possible positive integer which can be determined from the following upper bounds:*

- (i) *Let $m \geq 1$. The ordinary depth $d(H, G) \leq 2m + 1$ if and only if the distance between two irreducible characters of H is at most m .*
- (ii) *Let $m \geq 2$. The ordinary depth $d(H, G) \leq 2m$ if and only if $m(\chi) \leq m - 1$ for all $\chi \in \text{Irr}(G)$.*
- (iii) *$d(H, G) \leq 2$ if and only if H is normal in G , $d(H, G) = 1$ if and only if $G = HC_G(x)$ for all $x \in H$.*

Theorem 1.3. [2, Thm 6.9] *Suppose that H is a subgroup of a finite group G and $N = \text{Core}_G(H)$ is the intersection of m conjugates of H . Then H has ordinary depth $\leq 2m$ in G . If $N \leq Z(G)$ also holds, then $d(H, G) \leq 2m - 1$.*

A trivial consequence of Theorem 1.3 is that if G is simple and H is a proper subgroup having a disjoint conjugate, then $d(H, G) = 3$.

The paper is organized as follows. In Section 2 first we introduce the main information about Ree groups, after that in each section we determine the combinatorial and ordinary depth of a fixed maximal subgroup, more precisely: for $N_G(P)$ ($P \in \text{Syl}_3(G)$), $N_G(M^{\pm 1})$ ($M^{\pm 1} \in \text{Hall}_{q \pm 3m+1}(G)$), $N_G(M)$ ($M \in \text{Hall}_{\frac{q+1}{4}}(G)$), $C_G(i)$ ($o(i) = 2$), $G_0 \simeq R(q_0)$, $q_0 > 3$ and for $q_0 = 3$.

Our main result is the following.

Theorem 1.4. *Using the notations of Theorem 2.1, the combinatorial and the ordinary depths of the maximal subgroups in $G = R(q)$ for $q \geq 27$ are the following: $d_c(N_G(P), G) = d(N_G(P), G) = 5$, $d_c(N_G(M^1), G) = d_c(N_G(M^{-1}), G) = 4$, $d(N_G(M^1), G) = d(N_G(M^{-1}), G) = 3$, $d_c(N_G(M), G) = d_c(C_G(i), G) = 6$, $d(N_G(M), G) = d(C_G(i), G) = 3$, and $d_c(G_0, G) = 4$, $d(G_0, G) = 3$.*

2. GENERAL INFORMATION ON REE GROUPS

First let us recall some facts about Ree groups, ${}^2G_2(q) = R(q)$.

Theorem 2.1. [12, Ch XI. Thm. 13.2, p.292], [23], [19], [21, L. 2, L. 3] *Suppose that $q = 3^{2n+1}$ ($n \geq 1$), then there exist groups $G = R(q)$ of order $(q^3 + 1)q^3(q - 1)$ with the following properties:*

- (a) G is simple.
- (b) G is doubly transitive on $\Omega = \{1, \dots, q^3 + 1\}$. The stabilizer G_1 of point 1 is $N_G(P)$, where $P \in \text{Syl}_3(G)$. The stabilizer $G_{1,2}$ of two symbols is a cyclic group W of order $q - 1$ and $N_G(P) = PW$. We denote by $W_{2'}$ the 2-prime part of W , which is a Hall subgroup of order $(q - 1)/2$ and $N_G(W_{2'}) = N_G(W)$ is dihedral of order $2(q - 1)$. If j is the involution of W , and P' is the derived subgroup of P , then $C_P(j) = C_{P'}(j)$ is elementary abelian of order q and $C_P(j) \cap Z(P) = \{1\}$. If w is a nontrivial element of W of odd order, then $C_P(w) = \{1\}$. The subgroup fixing exactly 3 letters has order 2.
- (c) A Sylow 2-subgroup S of G is elementary abelian of order 8. Moreover, $C_G(S) = S$ and $|N_G(S)| = 8 \cdot 7 \cdot 3$. The 2-subgroups of equal order are conjugate in G .
- (d) G has cyclic Hall subgroups M^i ($i = \pm 1$) of orders $q + 1 + i \cdot 3m$, where $m = 3^n$. Each subgroup M^i is TI and it is the centralizer of each of its non-identity elements. The subgroups $B^i := N_G(M^i)$ are Frobenius groups with kernel M^i and cyclic complement of order 6.
- (e) For each subgroup V of order 4 there exists a cyclic Hall TI-subgroup M of order $\frac{q+1}{4}$ and an element t of order 6 such that $N_G(V) = N_G(M) = V \rtimes (M \rtimes \langle t \rangle) \simeq (V \times (M \rtimes C_2)) \rtimes C_3$, $C_G(M) = V \times M$ and $C_G(V) \simeq V \times (M \rtimes C_2)$.
- (f) For each nontrivial subgroup $H \leq A$, where A is one of the subgroups $W_{2'}$, M , $M^{\pm 1}$, the containment $N_G(H) \leq N_G(A)$ holds.
- (g) The centralizer in G of an involution j is isomorphic to $\langle j \rangle \times \text{PSL}_2(q)$.
- (h) A Sylow 3-subgroup P of G has order q^3 . It is a TI set. Its centre $Z(P)$ is an elementary abelian subgroup of order q , P is of class 3, and $P' = \Phi(P)$ is an elementary abelian subgroup of order q^2 containing $Z(P)$. The elements of $P \setminus P'$ have order 9, their cubes are forming $Z(P) \setminus \{1\}$.

P is isomorphic to the group of all triples (x, y, z) with $x, y, z \in GF(q)$ and the following multiplication rule:

$$(x_1, y_1, z_1)(x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2 + x_1(x_2\sigma), z_1 + z_2 - x_1y_2 + y_1x_2 - x_1(x_1\sigma)x_2).$$

Here σ is the automorphism of $GF(q)$ such that $x\sigma^2 = x^3$ for all $x \in GF(q)$. We have

$$P' = \Phi(P) = \{(0, y, z) \mid y, z \in GF(q)\} \text{ and } Z(P) = \{(0, 0, z) \mid z \in GF(q)\}$$

- (i) The maximal subgroups of G are up to conjugacy the following subgroups: $N_G(P)$, $C_G(j)$, $N_G(M^{\pm 1})$, $N_G(M)$, $G_0 \simeq R(q_0)$, where $q_0^a = q$ for some prime a . See [17, Theorem C], [19], [25, Ch 4.11.3 Thm 4.3].
- (j) $|G| = 2^3 3^{3(2n+1)} \frac{(q-1)}{2} \frac{(q+1)}{4} (q + 1 - 3^{n+1})(q + 1 + 3^{n+1})$, where any two of integers $2, 3, (q - 1)/2, (q + 1)/4, (q + 1 - 3^{n+1}), (q + 1 + 3^{n+1})$ are relatively prime. Each cyclic subgroup of G of order $\frac{q+1}{4}, q + 1 - 3^{n+1}$ and $q + 1 + 3^{n+1}$, respectively, can be embedded into a cyclic subgroup of order $\frac{q^3+1}{4} = \frac{q+1}{4}(q + 1 - 3^{n+1})(q + 1 + 3^{n+1})$.

In the following we will use the notation of Theorem 2.1. From Theorem 2.1 some further properties on the structure of G can be deduced.

- Corollary 2.2.** (1) All subgroups of order $\frac{q-1}{2}$, $\frac{q+1}{4}$, $q + 3^{n+1} + 1$, or $q - 3^{n+1} + 1$ are conjugate in G , respectively.
 (2) The subgroups of order $q - 1$ are conjugate in G .
 (3) The involutions of $N_G(P)$ are conjugate in $N_G(P)$.

We will need the following properties on centralizers of cyclic subgroups of G :

- Lemma 2.3.** (i) If $C \neq \{1\}$ is a cyclic subgroup of G , whose order divides $\frac{q-1}{2}$, then $C_G(C) \simeq C_{q-1}$.
 (ii) If $C \neq \{1\}$ is a cyclic subgroup of G , whose order divides $\frac{q+1}{4}$, then $C_G(C) \simeq C_2^2 \times C_{\frac{q+1}{4}}$.

Proof. (i) Let C be a nontrivial cyclic subgroup of G , whose order divides $\frac{q-1}{2}$. Since the subgroups of order $\frac{q-1}{2}$ are Hall-subgroups and cyclic, thus by a result of Wielandt [24], there is a cyclic subgroup C_{max} of order $\frac{q-1}{2}$ such that $C \leq C_{max}$. By Theorem 2.1, b) and f) $C_{q-1} \simeq C_G(C_{max}) \leq C_G(C) \leq N_G(C) \leq N_G(C_{max}) \simeq D_{2(q-1)}$. Thus $C_G(C) \simeq C_{q-1}$.

(ii) Similarly, if C is a nontrivial cyclic subgroup of G of order dividing $\frac{q+1}{4}$, then it is contained in a cyclic subgroup C_{max} of order $\frac{q+1}{4}$, and by Theorem 2.1 e) and f), $V \times C_{max} \simeq C_G(C_{max}) \leq C_G(C) \leq N_G(C) \leq N_G(C_{max}) \simeq (V \times (C_{max} \rtimes C_2) \rtimes C_3)$. However, by Theorem 2.1 c) $C_G(S) = S$, hence no 2-element in $N_G(C_{max}) \setminus V$ centralizes C . Since the Sylow 3-subgroups of G are TI, if a 3-element x would centralize $C = \langle m \rangle$, then m would normalize a Sylow 3-subgroup, contradicting that $|N_G(P)| = q^3(q-1)$, which is relatively prime to $\frac{q+1}{4}$. Hence $C_G(C) \simeq C_2^2 \times C_{\frac{q+1}{4}}$. \square

Using the representation of $P \in Syl_3(G)$ in Theorem 2.1 (h), we get the following useful results.

- Lemma 2.4.** If $p \in P' \setminus Z(P)$ for some $P \in Syl_3(G)$, then $C_P(p) = P'$.

Proof. Thus $p = (0, b, c)$ and $b \neq 0$. Then

$$(0, b, c)(x, y, z) = (x, b + y, c + z + bx) \text{ and } (x, y, z)(0, b, c) = (x, b + y, c + z - xb).$$

Therefore, $(x, y, z) \in C_P(p)$ if and only if $(x, y, z) \in P'$. \square

From this, Theorem 2.1 and from the character table of $N_G(P)$, see below or in [18], we deduce the following:

- Corollary 2.5.** (i) Let $z \in Z(P)$ be a nontrivial element, then $C_G(z) = P$.
 (ii) Let $y \in P' \setminus Z(P)$. The $C_G(y) = P' \rtimes \langle j \rangle$, where j is an involution.
 (iii) Let $x \in P \setminus P'$ then $C_G(x) = Z(P)\langle x \rangle$.

Lemma 2.6. Let $P \in Syl_3(G)$ and $(x, y, z), (a, b, c) \in P$. Then $(a, b, c)^{-1} = (-a, -b + a(a\sigma), -c)$ and $(x, y, z)^{(a, b, c)} = (x, y + x(a\sigma) - a(x\sigma), z - 2xb + 2ya - ax(x\sigma) + ax(a\sigma))$.

Proof. Check, that $(a, b, c)(a, b, c)^{-1} = (0, 0, 0)$:

$$(-a, -b + a(a\sigma), -c)(a, b, c) = (0, a(a\sigma) + (-a)(a\sigma), -(-a)b + (-b + a(a\sigma))a - (-a)(-a\sigma)a) = (0, 0, a^2(a\sigma) - a^2(a\sigma)) = (0, 0, 0).$$

$$\begin{aligned} (x, y, z)^{(a, b, c)} &= (-a, -b + a(a\sigma), -c)(x + a, y + b + x(a\sigma), z + c - xb + ya - x(x\sigma)a) = \\ &= (x, y + a(a\sigma) + x(a\sigma) - a((x + a)\sigma), z - xb + ya - x(x\sigma)a + a(y + b + x(a\sigma)) + (-b + a(a\sigma))(x + a) - a(a\sigma)(x + a)) = \\ &= (x, y + x(a\sigma) - a(x\sigma), z - xb + ya - x(x\sigma)a + ay + ab + ax(a\sigma) - bx - ba + ax(a\sigma) + a^2(a\sigma) - ax(a\sigma) + a^2(a\sigma)) = (x, y + x(a\sigma) - a(x\sigma), z - 2xb + 2ya - ax(x\sigma) + ax(a\sigma)) \end{aligned}$$

3. THE DEPTH OF $N_G(P)$

Proposition 3.1. If $\alpha, \beta, \gamma \in \{1, \dots, q^3 + 1\}$, then $G_{\alpha, \beta, \gamma}$ is isomorphic to C_2 or 1. Moreover, for every $\alpha, \beta \in \Omega$ there exist $\gamma, \delta \in \Omega$ such that $G_{\alpha, \beta, \gamma} \simeq C_2$ and $G_{\alpha, \beta, \delta} \simeq 1$.

Proof. Since G acts on $\{1, \dots, q^3 + 1\}$ doubly transitively, it is enough to show, that there are $\delta, \iota, \kappa, \lambda \in \Omega$ such that $G_{1, 2, \delta} = \{1\}$ and $|G_{\iota, \kappa, \lambda}| = 2$. We know the second statement from Theorem 2.1(b).

If $G_{1, 2, \gamma} \simeq C_2$, then there exists a symbol, δ such that $G_{1, 2, \gamma, \delta} = \{1\}$. Otherwise $G_{1, 2, \gamma, \delta} = G_{1, 2, \gamma}$ for every δ and so $\{1\} \neq G_{1, 2, \gamma} \subseteq Stab(1, \dots, q^3 + 1)$, which is a contradiction. Let us choose a δ such that $G_{1, 2, \gamma, \delta} = \{1\}$, therefore $G_{1, 2, \delta} \neq G_{1, 2, \gamma}$ and both are in $G_{1, 2} \simeq C_{q-1}$.

By Theorem 2.1(b) we know that $G_{1, 2, \delta} \simeq C_2$ or $\{1\}$. Since $G_{1, 2}$ contains one subgroup of order 2, which is $G_{1, 2, \gamma}$, thus $G_{1, 2, \delta}$ has to be $\{1\}$. \square

Proposition 3.2. The combinatorial depth of $N_G(P)$ is 5.

Proof. We will use the condition (2) in Definition 1.1 to prove, that $d_c(N_G(P), G) \leq 5$. By Theorem 2.1 (b) we have that $G_1 = N_G(P)$. Let us examine the following series of subgroups:

$$G_1 \geq G_1^{(x_1)} \geq G_1^{(x_1, x_2)} \geq G_1^{(x_1, x_2, x_3)}.$$

Now using the knowledge on the stabilizers, the group $G_1^{(x_1, x_2, x_3)} = G_{1, (1)x_1, (1)x_2, (1)x_3}$ could be isomorphic to $1, C_2, W$ or G_1 . Let us consider the 4 different cases. We will show, that in every case we can find elements, y_1 and y_2 such that $G_1^{(x_1, x_2, x_3)} = G_1^{(y_1, y_2)}$.

- (A) If $G_1^{(x_1, x_2, x_3)} = G_1$, then $G_1^{x_i} = G_1$ holds for every i . Then $y_1 = x_1$ and $y_2 = x_2$ is a good choice.
- (B) If $|G_1^{(x_1, x_2, x_3)}| = q - 1$, then two of the three containments in our series cannot be strict. Then let $y_1 := x_1$ and $y_2 := x_2$, if $G_1 \neq G_1^{(x_1)}$ or $G_1^{(x_1)} \neq G_1^{(x_1, x_2)}$, otherwise let $y_1 := x_1$ and $y_2 := x_3$.
- (C) If $|G_1^{(x_1, x_2, x_3)}| = 2$, then there is one equality in our series. Also there exists an index $i > 1$ such that $G_1^{(x_1, \dots, x_{i-1})} = G_1^{(x_1, \dots, x_i)}$ or $G_1 = G_1^{(x_1)}$. Let $\{y_1, y_2\} := \{x_1, x_2, x_3\} \setminus \{x_i\}$ in the first case and let $\{y_1, y_2\} := \{x_2, x_3\}$ in the second case.
- (D) If $G_1^{(x_1, x_2, x_3)} = \{1\}$, then by Proposition 3.1 there exist α, β such that $G_{1, \alpha, \beta} = \{1\}$. Since G is (doubly) transitive, there are elements $y_1, y_2 \in G$ such that $\alpha = (1)y_1$ and $\beta = (1)y_2$. Thus $G_1^{(y_1, y_2)} = \{1\}$.

Now we have to check, that $x_1 g x_1^{-1} = y_1 g y_1^{-1}$ for any $g \in G_1^{(x_1, x_2, x_3)}$. This is automatically true except for the case (C) when $G_1^{x_1} = G_1$. Let $G_1^{(x_1, x_2, x_3)} = \langle j \rangle \simeq C_2$ and $G_1^{x_1} = G_1$. If the originally chosen y_1, y_2 are not suitable, we modify them in the following way. Let us choose $z \in C_G(j) \setminus G_1$, which is possible. Let $y_1 = x_1 z$. Therefore x_1 and y_1 are acting in the same way on $G_1^{(x_1, x_2, x_3)}$. Since $y_1 \notin N_G(G_1) = G_1$, thus $G_1 \neq G_1^{y_1}$. We get, that the subgroup $G_1^{(y_1)} = G_{1, (1)y_1}$ is a stabilizer of two different points. By Proposition 3.1 we know that, there is a point δ such that $G_{1, (1)y_1, \delta} \simeq C_2$. Furthermore, $G_{1, (1)y_1, \delta} \subseteq G_{1, (1)y_1} \simeq W$ and $G_{1, (1)y_1}$ contains only one involution, j . If we choose an element y_2 such that $(1)y_2 = \delta$, then $G_1^{(y_1, y_2)} = G_{1, (1)y_1, \delta} = \langle j \rangle = G_1^{(x_1, x_2, x_3)}$, and we are done.

To prove that $d_c(N_G(P), G) > 4$ we will use condition (1) in Definition 1.1. Let β, γ such that $G_{1, \beta, \gamma} = 1$, which is possible by Proposition 3.1. Let $x_1, x_2 \in G$ such that $(1)x_1 = \beta$, $(1)x_2 = \gamma$. Since G is transitive, such x_1 and x_2 exist. Thus $G_1^{(x_1, x_2)} = G_{1, \beta, \gamma} = \{1\}$, so by Theorem 2.1 (b) there is no element y_1 such that $G_1^{(x_1, x_2)} = G_1^{(y_1)}$. This implies, that $d_c(N_G(P), G) = 5$. \square

Below we present the character table of $N_G(P)$ (see [18]). To shorten our notation, let $\zeta := 1 + i\sqrt{3}m$ and $\xi := \frac{1}{2}(m + \sqrt{3}m i)$.

The elements X, Y and T are fixed elements in $Z(P)$, $P \setminus P'$ and $P' \setminus Z(P)$, respectively. The element J is the involution in W and the element R is a generator of $W_{2'}$. Furthermore ϵ is a primitive root of unity of order $\frac{q-1}{2}$ and $a, b \in \mathbb{Z}_{\frac{q-3}{2}} \setminus \{0\}$.

	1	X	Y	T	T^{-1}	YT	YT^{-1}	JT	JT^{-1}	R^a	JR^a	J
$\mathbf{1}$	1	1	1	1	1	1	1	1	1	1	1	1
Δ	1	1	1	1	1	1	1	-1	-1	1	-1	-1
ψ_b^+	1	1	1	1	1	1	1	1	1	ϵ^{ab}	ϵ^{ab}	1
ψ_b^-	1	1	1	1	1	1	1	-1	-1	ϵ^{ab}	$-\epsilon^{ab}$	-1
α_1	$q-1$	$q-1$	-1	$q-1$	$q-1$	-1	-1	0	0	0	0	0
α_2	$(q-1)q$	$-q$	0	0	0	0	0	0	0	0	0	0
α_3	$m(q-1)$	$m(q-1)$	$-m$	$-m\bar{\zeta}$	$-m\zeta$	ξ	$\bar{\xi}$	0	0	0	0	0
α_4	$m(q-1)$	$m(q-1)$	$-m$	$-m\zeta$	$-m\bar{\zeta}$	$\bar{\xi}$	ξ	0	0	0	0	0
α_5	$\frac{m}{2}(q-1)$	$\frac{1}{2}m(q-1)$	m	$-\frac{1}{2}m\bar{\zeta}$	$-\frac{1}{2}m\zeta$	$-\xi$	$-\bar{\xi}$	$-\frac{1}{2}\bar{\zeta}$	$-\frac{1}{2}\zeta$	0	0	$\frac{q-1}{2}$
α_6	$\frac{m}{2}(q-1)$	$\frac{1}{2}m(q-1)$	m	$-\frac{1}{2}m\bar{\zeta}$	$-\frac{1}{2}m\zeta$	$-\xi$	$-\bar{\xi}$	$\frac{1}{2}\bar{\zeta}$	$\frac{1}{2}\zeta$	0	0	$-\frac{q-1}{2}$
α_7	$\frac{m}{2}(q-1)$	$\frac{1}{2}m(q-1)$	m	$-\frac{1}{2}m\bar{\zeta}$	$-\frac{1}{2}m\zeta$	$-\bar{\xi}$	$-\xi$	$-\frac{1}{2}\zeta$	$-\frac{1}{2}\bar{\zeta}$	0	0	$\frac{q-1}{2}$
α_8	$\frac{m}{2}(q-1)$	$\frac{1}{2}m(q-1)$	m	$-\frac{1}{2}m\bar{\zeta}$	$-\frac{1}{2}m\zeta$	$-\bar{\xi}$	$-\xi$	$\frac{1}{2}\zeta$	$\frac{1}{2}\bar{\zeta}$	0	0	$-\frac{q-1}{2}$

The centralizers of the element can be found in [18], using them we can get that $|Cl(X)| = q - 1$, $|Cl(Y)| = \frac{1}{3}q^2(q - 1)$, $|Cl(T)| = |Cl(T^{-1})| = \frac{1}{2}q(q - 1)$, $|Cl(YT)| = |Cl(YT^{-1})| = \frac{1}{3}q^2(q - 1)$, $|Cl(JT)| = |Cl(JT^{-1})| = \frac{1}{2}q^2(q - 1)$, $|Cl(R^a)| = |Cl(JR^a)| = q^3$, and $|C(J)| = q^2$.

To compute the induced characters $\mathbf{1}^G \Delta^G$ and ψ_b^{+G}, ψ_b^{-G} we need the following lemma.

- Lemma 3.3.** *a) Let $p \in P$. For an element $x \in G$, the element p^x is in $N_G(P)$ if and only if x is in $N_G(P)$.*
b) Let i be an involution in $N_G(P)$. For an element $x \in G$, the element i^x is in $N_G(P)$ if and only if x is in $C_G(i)N_G(P)$.
c) Let $w \in W \setminus \{i\}$. For an element $x \in G$, the element w^x is in $N_G(P)$ if and only if x is in $N_G(W)N_G(P)$.

- d) Let i be an involution in $N_G(P)$ and $p \in P$ such that $o(ip) = 6$. For an element $x \in G$, the element $(ip)^x$ is in $N_G(P)$ if and only if x is in $N_G(P)$.

Proof. a) Let assume that $p \in P, P^{x^{-1}}$. Since the Sylow 3-subgroups in G are TI, we get that $x \in N_G(P)$. The other direction is trivial.

b) Let assume that $i, i^x \in N_G(P)$. Since the involutions in $N_G(P)$ are conjugate, there is an element $n \in N_G(P)$ such that $(i^x)^n = i$. Thus $xn \in C_G(i)$, or equivalently $x \in C_G(i)N_G(P)$. The other direction is trivial.

c) Without loss of generality we can suppose that $w \in W_{2'}$ and $w^x \in N_G(P)$. Otherwise, we raise w into a suitable 2-power. Since $W_{2'}$ is a Hall subgroup of order $\frac{q-1}{2}$, by a result of Wielandt [24], we have that $W_{2'}$, $W_{2'}^x$ are conjugate in $N_G(P)$. Thus, $w, (w^x)^n \in W_{2'}$ for some $n \in N_G(P)$. Since $W_{2'}$ is cyclic, by Theorem 2.1 (f), and (b) we get that $xn \in N_G(\langle w \rangle) \leq N_G(W_{2'}) = N_G(W)$. Thus $x \in N_G(W)N_G(P)$. The other direction is trivial.

d) By the assumption, $(ip)^2 \in P$. On the other hand, $((ip)^2)^x \in N_G(P)$, thus by using one direction of Part a), we have that $x \in N_G(P)$. The other direction is trivial. \square

Using the previous Lemma we get the following

Corollary 3.4. *The characters $\mathbf{1}^G|_{N_G(P)}$, $\Delta^G|_{N_G(P)}$, $(\psi_b^+)^G|_{N_G(P)}$ and $(\psi_b^-)^G|_{N_G(P)}$ have the following values:*

$$\begin{array}{l|cccccccccccc} \mathbf{1}^G|_{N_G(P)} & q^3+1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & q+1 \\ \Delta^G|_{N_G(P)} & q^3+1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & 2 & -2 & -q-1 & \\ (\psi_b^+)^G|_{N_G(P)} & q^3+1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2\epsilon^{ab} & 2\epsilon^{ab} & q+1 & \\ (\psi_b^-)^G|_{N_G(P)} & q^3+1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & 2\epsilon^{ab} & -2\epsilon^{ab} & -q-1 & \end{array}$$

Proposition 3.5. *The ordinary depth of $N_G(P)$ is 5.*

Proof. We determined the irreducible constituents of $\mathbf{1}^G|_{N_G(P)} - 2(\mathbf{1}|_{N_G(P)})$, $\Delta^G|_{N_G(P)} - 2\Delta$, $(\psi_b^+)^G|_{N_G(P)} - 2\psi_b^+$ and $(\psi_b^-)^G|_{N_G(P)} - 2\psi_b^-$. Since these characters have zero values where the different ψ_b^+ , (or ψ_b^-) differ from each other, we get that the multiplicity of ψ_b^+ (or of ψ_b^-) in these four characters is independent of b' .

With the help of this, we determined the irreducible constituents of the characters in Corollary 3.4. The computed results are the following:

$$\begin{aligned} \mathbf{1}^G|_{N_G(P)} &= 2\mathbf{1}|_{N_G(P)} + \alpha_1 + q\alpha_2 + m\alpha_3 + m\alpha_4 + \frac{m+1}{2}\alpha_5 + \frac{m-1}{2}\alpha_2 + \frac{m+1}{2}\alpha_7 + \frac{m-1}{2}\alpha_8, \\ \Delta^G|_{N_G(P)} &= 2\Delta + \alpha_1 + q\alpha_2 + m\alpha_3 + m\alpha_4 + \frac{m-1}{2}\alpha_5 + \frac{m+1}{2}\alpha_6 + \frac{m-1}{2}\alpha_7 + \frac{m+1}{2}\alpha_8, \\ (\psi_b^+)^G|_{N_G(P)} &= 2\psi_b^+ + \alpha_1 + q\alpha_2 + m\alpha_3 + m\alpha_4 + \frac{m+1}{2}\alpha_5 + \frac{m-1}{2}\alpha_6 + \frac{m+1}{2}\alpha_7 + \frac{m-1}{2}\alpha_8, \\ (\psi_b^-)^G|_{N_G(P)} &= 2\psi_b^- + \alpha_1 + q\alpha_2 + m\alpha_3 + m\alpha_4 + \frac{m-1}{2}\alpha_5 + \frac{m+1}{2}\alpha_6 + \frac{m-1}{2}\alpha_7 + \frac{m+1}{2}\alpha_8. \end{aligned}$$

The distance between $\beta, \gamma \in \text{Irr}(N_G(P))$ is one in G if and only if $[\beta^G|_{N_G(P)}, \gamma] \neq 0$. Hence the distance between arbitrary elements of $\{\mathbf{1}, \Delta, \psi_b^+, \psi_b^-\}$ and arbitrary elements of $\{\alpha_i\}_{i=1}^8$ is one. In particular, the distance between two irreducible characters of $N_G(P)$ is at most 2. Thus by condition (i) in Definition 1.2 we get that $d(N_G(P), G) \leq 5$. Clearly $d(\mathbf{1}, \Delta) = 2$. Moreover, $m(\mathbf{1}_G) = \max_{\alpha \in \text{Irr}(N_G(P))} \min_{\chi \in \{\mathbf{1}|_{N_G(P)}\}} d(\alpha, \chi) = \max_{\alpha \in \text{Irr}(N_G(P))} d(\alpha, \mathbf{1}|_{N_G(P)}) = 2$. Thus by condition (ii) in Definition 1.2 we get that $d(N_G(P), G) = 5$. \square

4. THE DEPTH OF $N_G(M^1)$ AND $N_G(M^{-1})$

Proposition 4.1. *We have that $d_c(B^1, G) = d_c(B^{-1}, G) = 4$ and $d(B^1, G) = d(B^{-1}, G) = 3$*

Proof. We prove the statement for B^1 , the proof for B^{-1} is similar.

We use condition (1) in Definition 1.1 to prove $d_c(B^1, G) \leq 4$.

If $x_1 \in B^1$, then $(B^1)^{(x_1)} = B^1$. If $x_1 \notin B^1$, then $(B^1)^{(x_1)}$ is a subgroup of a Frobenius complement, since M^1 is TI and $B^1 = N_G(B^1) = N_G(M^1)$. Thus, the subgroup $(B^1)^{(x_1, x_2)}$ can be either B^1 or isomorphic to $\{1\}, C_2, C_3, C_6$.

(A) If the intersection is B^1 , then let $y_1 := x_1$, and we are done.

(B) Now we examine the case, when the intersection is $\{1\}$. We have to show, that there is an element y_1 such that $(B^1)^{(y_1)} = \{1\}$.

We will compute, how many elements y exist such that $(B^1)^{(y)}$ contains an involution or a 3-element.

- First we will determine at most how many y exist, such that $(B^1)^{(y)}$ contains an involution. We observe that:

If the involution i is an element of $B^1 \cap (B^1)^y$, then $y \in B^1 C_G(i)$.

To prove this: if $i \in B^1 \cap (B^1)^y$, then there exists an element $i_1 \in B^1$ such that $i_1^y = i$. Since i, i_1 are contained in some conjugates of the Frobenius complement of B^1 , there is an element $b \in M^1$ such that $i_1 = i^b$. Thus, $by \in C_G(i)$.

Let us estimate the number of elements y : $|\{y \mid \exists i \in (B^1)^{(y)}, o(i) = 2\}| \leq \sum_{\{i \in B^1 \mid o(i)=2\}} |B^1 C_G(i)| \leq |M^1| |B^1 C_G(i)| = (q+1+3m)^2(q-1)(q+1)q$.

- Now we will determine, how many elements y exist, such that $(B^1)^{(y)}$ contains a 3-element. We observe that:

If $T \subseteq B^1 \cap (B^1)^y$ and $|T| = 3$, then $y \in B^1 N_G(T)$.

To prove this let us assume that $T \subseteq B^1 \cap (B^1)^y$. Then there is a subgroup $T_1 \leq B^1$ such that $T_1^y = T$. Since T_1 and T are contained in some conjugates of the Frobenius complement of B^1 , there exists an element $b \in M^1$ such that $T_1^b = T$. Therefore $b^{-1}y \in N_G(T)$ and $y \in B^1 N_G(T)$.

Since every element of $N_G(T)$ normalizes the Sylow 3-subgroup containing T , thus

$|N_G(T)| \mid |N_G(P)| = q^3(q-1)$. By Cor. 2.5 (ii), $|C_G(T)| = 2q^2$. Since $4 \nmid q-1$, we get that $|N_G(T)| = 2q^2$. We have that: $|\{y \mid 3 \mid |B^1 \cap (B^1)^y|\}| \leq |\cup_{\{T \subseteq B^1 \mid |T|=3\}} B^1 N_G(T)| \leq |M^1| |B^1 N_G(T)| = 2(q+1+3m)^2 q^2$.

Now some element y must exist such that $(B^1)^{(y)} = \{1\}$, since: $|\{y \mid (B^1)^{(y)} = 1\}| \geq |G| - |\{y \mid \exists i : i \in (B^1)^{(y)}, o(i) = 2\}| - |\{y \mid \exists T : T \leq (B^1)^{(y)}, |T| = 3\}| \geq q^3(q^3+1)(q-1) - (q+1+3m)^2(q-1)(q+1)q - 2(q+1+3m)^2 q^2 > 0$.

- (C) Now we can assume that the intersection is nontrivial and cyclic.

Let $x'_i := \{x_1, x_2\} \setminus x_i$. If $x_i \in \{x_1, x_2\}$ and it satisfies $(B^1)^{(x_i)} = B^1$, then $(B^1)^{(x_1, x_2)} = (B^1)^{(x'_i)}$ holds. Otherwise, both $(B^1)^{(x_1)}$ and $(B^1)^{(x_2)}$ are subgroups of some Frobenius complements of B^1 . Since the Frobenius complements are TI sets, the intersection of $(B^1)^{(x_1)}$ and $(B^1)^{(x_2)}$ is either trivial, or one of them contains the other, and the intersection is the smaller one.

Since in every case, (A)-(C), we can choose an element y such that $(B^1)^{(x_1, x_2)} = (B^1)^{(y)}$, we have that $d_c(B^1, G) \leq 4$.

Now we will show an example, where there is no convenient y , which can act in the same way as x_1 on the intersection. Let i be an involution in B^1 and let us choose an element $x_1 \in M^1 \setminus C_G(i)$. Let $x_2 \in C_G(i) \setminus B^1$. Thus $i \in (B^1)^{(x_1, x_2)} \neq B^1$. Let us suppose by contradiction that an element y is suitable: $(B^1)^{(x_1, x_2)} = (B^1)^{(y)}$ and $i^{x_1} = i^y$ hold.

Thus $i \in (B^1)^y$ and hence $i_1 = i^{y^{-1}} \in B_1$. Since the product of two involutions of B^1 is in M^1 , thus $(i_1 i)^y \in (M^1)^y$. On the other hand, $(i_1 i)^y = i i^y = i i^{x_1} \in M^1$. Since M^1 is TI and $y \notin B^1 = N_G(M^1)$, thus $(i_1 i)^y = i i^y \in (M^1)^y \cap M^1 = \{1\}$. Therefore, $y \in C_G(i)$, which is a contradiction, since $x_1 \notin C_G(i)$, however $i^y = i^{x_1}$.

Thus $d_c(B^1, G) = 4$ and similarly $d_c(B^{-1}, G) = 4$.

We have seen that there exist elements $x_1, x_2 \in G$ such that $(B^1)^{(x_1)} = (B^{-1})^{(x_2)} = \{1\}$. Since G is simple, by Theorem 1.3 we get that $d(B^1, G) = d(B^{-1}, G) = 3$.

□

5. THE DEPTH OF $N_G(M)$

In this section M will be a fixed cyclic subgroup of G of order $\frac{q+1}{4}$. Let V be a Klein subgroup commuting with M . We know, that $C_G(M) = V \times M$ and $N_G(M) = N_G(V) \simeq V \rtimes (M \rtimes C_6) \simeq (V \times (M \rtimes C_2)) \rtimes C_3$, and $C_G(V) \simeq V \times (M \rtimes C_2)$

Lemma 5.1. *If there is a nontrivial element $m \in N_G(M)^{(x_1)}$, whose order divides $\frac{q+1}{4}$, then $x_1 \in N_G(M)$ i.e. $N_G(M)^{(x_1)} = N_G(M)$.*

Proof. We know that $N_G(M) \simeq (C_2^2 \times (M \rtimes C_2)) \rtimes C_3$. This means, that m has to be in $M \cap M^{x_1}$. Therefore, by Lemma 2.3, we have that $M, M^{x_1} \leq C_G(m) \simeq C_{\frac{q+1}{4}} \times C_2^2$ and hence $M = M^{x_1}$. □

Proposition 5.2. *Let V be as above. Then $\mathcal{U}_1(N_G(M)) = \{N_G(M) = N_G(V), S, H \mid V \leq S \in \text{Syl}_2(G), [H, V] = 1, V \neq H \simeq C_2^2\} \cup U$, where $U \subseteq \{H \mid H \simeq \{1\}, H \simeq C_2, C_3 \text{ or } C_6\}$.*

Proof. We know from Theorem 2.1 (c) that the Klein subgroups of G are conjugate.

By Lemma 5.1 we have that $N_G(M)^{(x_1)}$ is isomorphic to one of the following subgroups:

$1, C_2, C_3, C_2^2, C_6, C_2^3, (C_2^2) \rtimes C_3, (C_2^3) \rtimes C_3, N_G(M)$.

It is obvious that $N_G(M)$ occurs. Let us examine the other cases.

- (A) $C_2^3 \lesssim N_G(M)^{(x_1)}$: **We will prove, that if $N_G(M)^{(x_1)}$ is a proper subgroup of $N_G(M)$ and contains a subgroup isomorphic to C_2^3 , then V is contained in $N_G(M)^{(x_1)}$ and $N_G(M)^{(x_1)}$**

is isomorphic to C_2^3 . Let S be a subgroup of $N_G(M)^{(x_1)}$ of order 8 and assume that $x_1 \notin N_G(M)$. Then $S \in \text{Syl}_2(G)$ and since $N_G(M) = N_G(V)$, we have that $V, V^{x_1} \leq S$. Thus V and V^{x_1} contain exactly one common involution, which we denote by i . Suppose, that $h \in N_G(M)^{(x_1)}$ is an element of order 3. Then obviously h acts on V and V^{x_1} nontrivially, so the action is transitive on the involutions. This is a contradiction, since i is the unique common involution of V and V^{x_1} . Thus, by Lemma 5.1, $N_G(M)^{(x_1)}$ can contain only 2-elements, hence $S = N_G(M)^{(x_1)}$.

Construction: Let $S \in \text{Syl}_2(G)$ containing V . We want to show that there exists an element $x_1 \notin N_G(M) = N_G(V)$ such that $N_G(M)^{(x_1)} = S$. Choose an element $x_1 \in G$ such that $V \neq V^{x_1} \leq S$. Then $S \leq N_G(V)^{(x_1)} = N_G(M)^{(x_1)}$ and we are done due to the previous statement.

(B) $C_2^2 \lesssim N_G(M)^{(x_1)}$: **We will prove, that if $N_G(M)^{(x_1)} \neq N_G(M)$ and it contains a Klein subgroup H , but it does not contain a Sylow 2-subgroup of G , then $N_G(M)^{(x_1)} = H$ and $[H, V] = 1$, moreover $V \neq H$.**

By the assumption and Lemma 5.1 $N_G(M)^{(x_1)}$ can only be isomorphic to C_2^2 and $C_2^2 \rtimes C_3$. Suppose now that $N_G(M)^{(x_1)} = H \rtimes \langle h \rangle$, where $H \simeq C_2^2$ and $o(h) = 3$. Thus h acts on H, V, V^{x_1} nontrivially.

Since $H \leq N_G(V) = N_G(M)$, $H = V$ or $\langle H, V \rangle \in \text{Syl}_2(G)$. Thus $H = V$, because otherwise h could not act nontrivially on both H and V . Furthermore, $H = V^{x_1}$ due to the same explanation, which is a contradiction, since $x_1 \notin N_G(V)$. Thus $N_G(M)^{(x_1)} \not\cong C_2^2 \rtimes C_3$.

Now we prove that $N_G(M)^{(x_1)}$ cannot be V . If $V \leq N_G(M)^{(x_1)} = N_G(V)^{(x_1)}$, then $C_2^3 \simeq \langle V, V^{x_1} \rangle \leq N_G(V)^{(x_1)} = N_G(M)^{(x_1)}$, which is a contradiction.

Construction: Let H be a Klein subgroup of G different from V , with $[H, V] = 1$. Then $\langle H, V \rangle \in \text{Syl}_2(G)$. Let $H \cap V = \langle i \rangle$, and let j be another generator of V . Then $V \leq C_G(H) = H \times D$, where $D \simeq D_{\frac{q+1}{2}}$. We may assume that $j \in D$, otherwise we choose the complement $\langle C_{\frac{q+1}{4}}, j \rangle$ instead of D . Let $k \in D$ be an involution different from j . Choose an element $x_1 \in G$ such that $V^{x_1} = \langle i, k \rangle$. Then $[V, V^{x_1}] \neq 1$. The subgroup $N_G(V)^{(x_1)} = N_G(M)^{(x_1)}$ cannot contain a Sylow 2-subgroup S of G , otherwise $V, V^{x_1} \leq S$. However, it contains H . Hence $N_G(M)^{(x_1)} = H$.

(C) $1, C_2, C_3, C_6$: $N_G(M)^{(x_1)}$ can be isomorphic to $1, C_2, C_3$ or C_6 . To compute the depth of $N_G(V)$, however, we do not need to determine exactly which subgroups can occur. □

Lemma 5.3. $\mathcal{U}_2(N_G(M)) = \{1, C, K, S, N_G(M) = N_G(V) \mid C \simeq C_2, K \simeq C_2^2, S \simeq C_2^3 \text{ and each of them commutes with } V\} \cup \{X \cap Y \mid X, Y \in \mathcal{U}\}$, where \mathcal{U} is as before. Thus $\mathcal{U}_1(N_G(M)) \neq \mathcal{U}_2(N_G(M))$, since V is in the difference set. Hence we have that $d_c(N_G(M), G) \geq 5$.

Proof. If $Z \in \mathcal{U}_2(N_G(M))$, then $Z = X \cap Y$ where $X, Y \in \mathcal{U}_1(N_G(M))$. Therefore we will study the intersection of subgroups of different structure from $\mathcal{U}_1(N_G(M))$ in a $\mathcal{U}_1 - \mathcal{U}_1$ table. The elements of first the row will be the possible structure of X and similarly the elements of the first column will be the possible structure of Y . These can be $N_G(M)$; a Sylow 2-subgroup S containing V ; a Klein subgroup H , which commutes with V , but not equal to it; some subgroups isomorphic to C_6, C_3, C_2 or 1 (if they occur at all, we denote them by Z_6, Z_3, Z_2, Z_1). The element in the position (X, Y) in the table will show, which structure can occur as $X \cap Y$.

	$N_G(M)$	S	H	Z_6	Z_3	Z_2	Z_1
$N_G(M)$	$N_G(M)$	S	H	Z_6	Z_3	Z_2	1
S	S	S, V	H, C_2	$C_2, 1$	1	$Z_2, 1$	1
H	H	H, C_2	$H, C_2, 1$	$C_2, 1$	1	$Z_2, 1$	1
Z_6	Z_6	$C_2, 1$	$C_2, 1$				
Z_3	Z_3	1	1				
Z_2	Z_2	$Z_2, 1$	$Z_2, 1$	$X \cap Y, \text{ where } X, Y \in \mathcal{U}$			
Z_1	1	1	1				

It is obvious that V occurs as the intersection of two Sylow 2-subgroups containing it. Thus $V \in \mathcal{U}_2(N_G(M))$. The other Klein subgroups, which commute with V , occur already in $\mathcal{U}_1(N_G(M))$. To finish the proof, we have to show, that every cyclic subgroup of order 2, which commutes with V , and also $\{1\}$ occur in $\mathcal{U}_2(N_G(M))$.

A subgroup of order 2 in V occurs in $\mathcal{U}_2(N_G(M))$, since we can choose two different Klein subgroups which are not equal to V , and commuting with it, such that their intersection is this subgroup. Let $\langle k \rangle$ be a subgroup of order 2, which commutes with V but not contained in it. Let $V = \langle i, j \rangle$. Then $K_1 = \langle i, k \rangle$ and $K_2 = \langle j, k \rangle$ both are in $\mathcal{U}_1(N_G(M))$. Thus their intersection $\langle k \rangle$ is in $\mathcal{U}_2(N_G(M))$. To see that $\{1\} \in \mathcal{U}_2$, we take two different Sylow 2-subgroups S_1, S_2 containing V . Let $V = \langle i, j \rangle$,

and let $K_1 \leq S_1$ be of order 4 with $K_1 \cap V = \langle i \rangle$. Let $K_2 \leq S_2$ be of order 4 with $K_2 \cap V = \langle j \rangle$. Then $K_1, K_2 \in \mathcal{U}_1(N_G(M))$, thus $\{1\} = K_1 \cap K_2 \in \mathcal{U}_2(N_G(M))$. \square

We remark, that every cyclic subgroup of order 6, which is contained in $\mathcal{U}_2(N_G(M))$, is already in $\mathcal{U}_1(N_G(M))$.

Lemma 5.4. $\mathcal{U}_2(N_G(M)) = \mathcal{U}_3(N_G(M))$. Thus, $d_c(N_G(M), G) \leq 6$.

Proof. Similarly to the previous case, if Z in $\mathcal{U}_3(N_G(M))$, then $Z = X \cap Y$, where $X \in \mathcal{U}_2(N_G(M))$, $Y \in \mathcal{U}_1(N_G(M))$. We demonstrate the possible intersections in a $\mathcal{U}_1 - \mathcal{U}_2$ table. The first column shows the possible values Y of $\mathcal{U}_1(N_G(M))$, as above. The first row shows the possible values X of $\mathcal{U}_2(N_G(M))$. They can be $N_G(M)$, the 2-subgroups, which commute with V : C_2^3 , C_2^2 , C_2 ; 1; and maybe some subgroups, which are isomorphic to C_6 or C_3 , denoted by ζ_6 and ζ_3 . Note that the 2-subgroup in Z_6 commutes with V . We have seen above that $\zeta_6 = Z_6$.

	$N_G(M)$	C_2^3	C_2^2	C_2	1	Z_6	ζ_3
$N_G(M)$	$N_G(M)$	C_2^3	C_2^2	C_2	1	Z_6	ζ_3
S	C_2^3	C_2^3, C_2^2	C_2^2, C_2	$C_2, 1$	1	$C_2, 1$	1
H	C_2^2	C_2^2, C_2	$C_2^2, C_2, 1$	$C_2, 1$	1	$C_2, 1$	1
Z_6	Z_6	$C_2, 1$	$C_2, 1$	$C_2, 1$	1	$Z_6, \zeta_3, C_2, 1$	$\zeta_3, 1$
Z_3	Z_3	1	1	1	1	$Z_3, 1$	$Z_3, 1$
Z_2	C_2	$C_2, 1$	$C_2, 1$	$C_2, 1$	1	$C_2, 1$	1
Z_1	1	1	1	1	1	1	1

The only new intersection could be isomorphic to C_3 , if X and Y isomorphic to cyclic subgroups of order 6. In this case both X and Y are in $\mathcal{U}_1(N_G(M))$ and hence the intersection is in $\mathcal{U}_2(N_G(M))$. \square

Theorem 5.5. $d_c(N_G(M), G) = 6$

Proof. We have seen that $5 \leq d_c(N_G(M), G) \leq 6$. We will give elements x_1, x_2, x_3 in G , such that one cannot find any $y_1, y_2 \in G$ such that $N_G(M)^{(x_1, x_2, x_3)} = N_G(M)^{(y_1, y_2)}$ and the elements x_1, y_1 act on $N_G(M)^{(x_1, x_2, x_3)}$ in the same way. We know that $C_G(V) = V \times (M \rtimes \langle t \rangle)$, where $M = \langle a \rangle$, $o(a) = \frac{q+1}{4}$ and $o(t) = 2$. Let $x_1 \in N_G(M)$ and let x_2, x_3 be such that $N_G(M)^{(x_2)} = V \times \langle t \rangle$ and $N_G(M)^{(x_3)} = V \times \langle ta \rangle$, which is possible by Proposition 5.2. Hence we get that $N_G(M)^{(x_1, x_2, x_3)} = N_G(M) \cap (V \times \langle t \rangle) \cap (V \times \langle ta \rangle) = V$.

Let us choose elements y_1, y_2 such that $N_G(M)^{(y_1, y_2)} = V$. Since x_1 normalizes V , if y_1 acts in the same way as x_1 , then y_1 also normalizes V . Therefore, $V = N_G(M) \cap N_G(M)^{y_1} \cap N_G(M)^{y_2} = N_G(M)^{(y_2)}$, which is a contradiction by Proposition 5.2. Thus $d_c(N_G(M), G) = 6$. \square

Proposition 5.6. *There is an element $x \in G$ such that $N_G(M)^{(x)} = \{1\}$.*

Proof. We need to estimate the number of elements x , such that $N_G(M)^x$ contains special elements: elements of M , 3-elements of $N_G(M)$, involutions of $N_G(M)$, respectively. First we make the following observations:

- We have seen in Lemma 5.1 that if $N_G(M)^x$ contains some nontrivial elements of M for some $x \in G$, then $x \in N_G(M)$. Thus we get that $A_{\frac{q+1}{4}} := \{x \in G \mid N_G(M)^{(x)} \cap M \neq 1\} = N_G(M)$.
- Let p be a nontrivial 3-element in $N_G(M)$. First we show that for an element $x \in G$ the subgroup $N_G(M)^x$ contains $\langle p \rangle$ if and only if $x \in N_G(M)N_G(\langle p \rangle)$.
Let $\langle p \rangle \leq N_G(M)^x$. Since the subgroups of order 3 are conjugate in $N_G(M)$, there is an element $n \in N_G(M)$ such that $\langle p \rangle^{x^{-1}} = \langle p \rangle^n$. Hence $nx \in N_G(\langle p \rangle)$, and $x \in N_G(M)N_G(\langle p \rangle)$. The other direction is trivial. Using that $N_G(\langle p \rangle) = C_G(\langle p \rangle)$ is of order $2q^2$ by Cor.2.5 (ii), we have: $|A_3| := |\{x \in G \mid N_G(M)^{(x)} \text{ contains 3-elements}\}| = |\cup_{\langle p \rangle \leq N_G(M)^{(x)} \text{ } o(p)=3} N_G(M)N_G(\langle p \rangle)| \leq \sum_{\langle p \rangle \leq N_G(M)^{(x)} \text{ } o(p)=3} \frac{6(q+1)2q^2}{6} = 2q^2(q+1)^2$.

- There are $|C_G(i)| = q(q-1)(q+1)$ elements in G , which take a fixed involution to a fixed involution. Furthermore, the subgroup $N_G(M) = (V \times (M \rtimes C_2)) \rtimes C_3$ contains $q+4$ involutions. Thus we have: $|A_2| := |\{x \in G \mid N_G(M)^{(x)} \text{ contains involutions}\}| \leq (q+4)^2 q(q-1)(q+1)$.

Using all the above results we have: $|\{x \in G \mid N_G(M)^{(x)} = 1\}| \geq |G| - |A_{\frac{q+1}{4}}| - |A_3| - |A_2| = q^3(q^3+1)(q-1) - 6(q+1) - 2q^2(q+1)^2 - (q+4)^2 q(q-1)(q+1)$. Since this is bigger than 1, we get that there is an element $x \in G$ such that $N_G(M)^{(x)} = \{1\}$. \square

Using Theorem 1.3 we have the following.

Corollary 5.7. *The ordinary depth of $N_G(M)$ is 3.*

6. THE DEPTH OF $C_G(i)$

Since $C_G(i) = \langle i \rangle \times L$, where $L \simeq PSL(2, q)$, the following theorem will be useful.

Theorem 6.1. (Dickson)[10, 8.27 Hauptsatz, Ch II, p. 213] *The complete list of all subgroups of $L = PSL(2, q)$ for $q = 3^{2n+1}$ is the following:*

- a) alternating groups A_4 ;
- b) elementary-abelian 3-groups of order at most q ;
- c) semidirect products $C_3^m \rtimes C_t$ of elementary-abelian groups of order at most q with cyclic groups of order t , where t divides $3^m - 1$ as well as $\frac{q-1}{2}$;
- d) groups $PSL(2, 3^m)$, if $m \mid 2n+1$;
- e) cyclic groups of order z , where z divides $\frac{q-1}{2}$;
- f) dihedral subgroups of order $2z$ with z as previous;
- g) cyclic groups of order z , where z divides $\frac{q+1}{2}$;
- h) dihedral subgroups of order $2z$ with z as previous.

Proposition 6.2. $\mathcal{U}_1(C_G(i)) = \{C_G(i), C_P(i), C_G(V), K, C, 1 \mid i \in N_G(P), P \in Syl_3(G), i \in V \simeq C_2^2, K \simeq C_2^2, [i, K] = 1, i \notin K, C \simeq C_2, [i, C] = 1, i \notin C\}$

Proof. We will examine what kind of subgroups $H \in \mathcal{U}_1(C_G(i))$ occur. If we find a subgroup type which could occur, we will also construct it. For this, we choose an involution j such that $C_G(i) \cap C_G(j) = H$. Since the involutions are conjugate in G , we will be done.

If $[i, i^{x_1}] = 1$, then $C_G(i)^{(x_1)} = C_G(i) \cap C_G(i^{x_1}) = C_G(V)$, where $V = \langle i, i^{x_1} \rangle$ is a Klein subgroup of G . By Theorem 2.1 (e), we have that $C_G(i) \cap C_G(i^{x_1}) \simeq (C_2^2 \times D_{\frac{q+1}{2}})$. We remark, that $i \in V = Z(C_G(i)^{(x_1)})$.

$C_G(V)$: Now we show that every subgroup $H \simeq C_2^2 \times D_{\frac{q+1}{2}}$, where $i \in Z(H)$, occurs in $\mathcal{U}_1(C_G(i))$. Let j be another involution in $Z(H)$. Thus $\langle i, j \rangle = Z(H)$ and $C_G(Z(H)) = C_G(i) \cap C_G(j) = H$ by Theorem 2.1 (e).

If $[i, i^{x_1}] \neq 1$, then let $D := \langle i, i^{x_1} \rangle$. Since $C_G(i) = \langle i \rangle \times L$, where $L \simeq PSL(2, q)$, therefore the projection to the second component $\pi_2(C_G(i)^{(x_1)}) = \pi_2(C_G(D))$ is a subgroup of $L \simeq PSL_2(q)$. Now we examine, which subgroups of $PSL_2(q)$ can occur. We study the subgroups of the list in Theorem 6.1 if they could be equal to $\pi_2(C_G(D))$.

- a)-b) **Let $[i, i^{x_1}] \neq 1$, and let $D = \langle i, i^{x_1} \rangle$ and let us suppose that $\pi_2(C_G(D))$ contains an element p of order 3. Then $C_G(D)$ also contains an element of order 3, moreover, $C_G(i)^{(x_1)} = C_P(i) \simeq C_3^{2n+1}$ for some $P \in Syl_3(G)$ with $i \in N_G(P)$. On the other hand, every subgroup of the form $C_P(i)$ occurs as $C_G(i)^{(x_1)}$ if $i \in N_G(P)$ for some $P \in Syl_3(G)$.**

By raising to the second power if necessary, we may assume that $p \in C_G(D)$. Let $p \in P \in Syl_3(G)$. We remark that, by Cor. 2.5, $p \in P \setminus Z(P)$. Since the Sylow 3-subgroups are TI, $i, i^{x_1} \in N_G(P)$. In particular, $i, i^{x_1} \in P \rtimes \langle i \rangle$. Thus $i^{x_1} = p'i$ for a suitable $p' \in P$. Since $[p, i] = 1$ and $[p, p'i] = 1$, thus $[p, p'] = 1$ also holds. Hence $p' \in P'$ by Lemma 2.4. If $p' \in Z(P)$, then by Cor.2.5 we have that $C_G(p') = P$. Thus $C_G(D) = C_G(i) \cap C_G(p') = C_P(i)$.

If $p' \notin Z(P)$, then by Cor. 2.5, $C_G(p') = P' \rtimes \langle j \rangle$, where j is an involution in $N_G(P)$. Since $[p', i] \neq 1$, we have that $j \neq i$. Hence $j = p_1 i$ for some $1 \neq p_1 \in P$. Thus $[j, i] \neq 1$. Hence $C_G(D) = C_G(i) \cap C_G(p') = C_{P'}(i) = C_P(i)$, by Theorem 2.1 (b).

$C_P(i)$: Now we will show, that every subgroup $C_P(i)$ occurs as $C_G(D)$, if $P \in Syl_3(G)$ and $i \in N_G(P)$. Let $P \in Syl_3(G)$ such that $i \in N_G(P)$. The involution i acts on $P' \simeq C_3^{4n+2}$ nontrivially. Choose an element $p \in P'$, which is not centralized by i . Let $p' := [p, i]$. Then $p' \in P'$ and it is inverted by i . Moreover, $j := p'i$ is an involution such that $C_G(\langle i, j \rangle) = C_G(\langle i, p' \rangle) = C_G(i) \cap C_G(p') = C_P(i) \simeq C_3^{2n+1}$, as above.

- c)-f) **Let $[i, i^{x_1}] \neq 1$ and let $D := \langle i, i^{x_1} \rangle$. Then $\pi_2(C_G(i)^{(x_1)}) = \pi_2(C_G(D))$ does not contain a nontrivial cyclic subgroup C , whose order divides $\frac{q-1}{2}$.** Raising to the second power if necessary, we may suppose that $C_G(D) \geq C$. Then $D \leq C_G(C) \simeq C_{q-1}$ by Lemma 2.3, which is a contradiction.

- g)-h) **Let $[i, i^{x_1}] \neq 1$, and let $D = \langle i, i^{x_1} \rangle$. Then if $\pi_2(C_G(i)^{(x_1)}) = \pi_2(C_G(D))$ is a dihedral subgroup of order $2z$ or a cyclic subgroup of order z , where $z \mid \frac{q+1}{2}$, then $z = 2$. Only C_2 and C_2^2 can occur as $C_G(D)$, (C_2^3 cannot occur). Obviously $i \notin C_G(D)$ and $[C_G(D), i] = 1$.**

If $z > 2$, then $\pi_2(C_G(D))$ contains a cyclic subgroup $C \neq \{1\}$ whose size divides $\frac{q+1}{4}$. Raising to the second power the elements of C , if necessary, we may assume that $C \leq C_G(D)$. Thus, by Lemma 2.3, $D \leq C_G(C) \simeq C_2^2 \times C_{\frac{q+1}{4}}$, which is a contradiction. Furthermore, if $S \in Syl_2(G)$ and $C_G(D) \geq S$ then $i, i^{x_1} \in S$, which is a contradiction.

We will show, that both C_2 and C_2^2 occur as $C_G(D)$ if $i \notin C_2^2$ and $[C_2^2, i] = 1$, or $i \notin C_2$ and $[C_2, i] = 1$.

C_2 : Let $H = \langle k \rangle \simeq C_2$ such that $k \neq i$ and $[k, i] = 1$. Then $i \in C_G(k) = \langle k \rangle \times U$, where $U \simeq PSL(2, q)$. Since the involutions in $PSL(2, q)$ are conjugate, see [10, 8.5 Satz, Ch. II, p. 193], if $i \in U$, then there exists a subgroup $T (\simeq D_{q-1}) \leq U$ such that $i \in T$. Let $C \leq T$ be the unique cyclic subgroup of order $\frac{q-1}{2}$ and let j be another involution in T . Then, by Lemma 2.3, $C_G(ij) \simeq C_{q-1}$, thus $C_G(ij) = \langle k \rangle \times C$. Thus, $C_G(i) \cap C_G(j) = C_G(i) \cap C_G(ij) = \langle k \rangle$. If $i \notin U$, then $i = ki_2$, and i_2 is in a dihedral subgroup $T \leq U$ of order $q-1$. Let $C \leq T$ be the unique cyclic subgroup of order $\frac{q-1}{2}$ and let $T_1 := \langle C, i \rangle$. Then T_1 is also dihedral of order $q-1$. Let j be another involution in T_1 . Then $C_G(ij) = \langle k \rangle \times C$. As above, we have that $C_G(i) \cap C_G(j) = \langle k \rangle$.

C_2^2 : Let $H \simeq C_2^2$ such that $i \notin H$, and $[i, H] = 1$. Then, by Theorem 2.1 (e), $i \in C_G(H) = H \times T \simeq C_2^2 \times D_{\frac{q+1}{2}}$. Let $C \leq T$ be the unique cyclic subgroup of order $\frac{q+1}{4}$. If $i \in T$ then let j be another involution in T . By Lemma 2.3, $C_G(ij) \simeq C_2^2 \times C_{\frac{q+1}{4}}$, thus $C_G(ij) = H \times C$. Hence, $C_G(i) \cap C_G(j) = C_G(i) \cap C_G(ij) = H$. If $i \notin T$, then $i = i_1 i_2$, where $i_1 \in H$ and $i_2 \in T$. Then let C be as above and let $T_1 := \langle i, C \rangle$. Then T_1 is also dihedral of order $\frac{q+1}{2}$. Let j be another involution in T_1 . Then $C_G(ij) = H \times C$. And hence $C_G(i) \cap C_G(j) = C_G(i) \cap C_G(ij) = H$.

Finally we show that the trivial subgroup also occurs as intersection.

1 : There is an involution j such that $C_G(i) \cap C_G(j) = \{1\}$

Let M^1 be as in Theorem 2.1 (d). Let M^3 be a conjugate of M^1 , which satisfies, that $i \in N_G(M^3)$ and let j be another involution in $N_G(M^3)$. Since $N_G(M^3)$ is a Frobenius group with kernel M^3 , we have that $\langle i, j \rangle$ is a dihedral subgroup of $N_G(M^3)$ and ij is an element of M^3 . Using Theorem 2.1 (d) and the fact, that $2|PSL(2, q)|$ and $q+3m+1$ are relatively prime, we get that $C_G(i) \cap C_G(j) = C_G(i) \cap C_G(ij) \simeq (\langle i \rangle \times PSL(2, q)) \cap C_{q+3m+1} = \{1\}$.

Hence $C_G(i)^{(x_1)}$ can have exactly the values mentioned in the Proposition. \square

Proposition 6.3. $\mathcal{U}_2(C_G(i)) = \mathcal{U}_1(C_G(i)) \cup \{\langle i \rangle\} \cup \{V \mid i \in V \simeq C_2^2\} \cup \{S \mid i \in S \simeq C_2^3\}$

Proof. We have seen in Theorem 6.2, that $\mathcal{U}_1(C_G(i)) = \{H_2 \mid H_2 \simeq C_2, i \notin H_2, [i, H_2] = 1\} \cup \{H_4 \mid H_4 \simeq C_2^2, i \notin H_4, [i, H_4] = 1\} \cup \{H_{2q+2} \mid H_{2q+2} \simeq C_2^2 \times D_{\frac{q+1}{2}}, i \in Z(H_{2q+2})\} \cup \{1, C_G(i)\} \cup \{C_P(i) \mid P \in Syl_3(G), i \in N_G(P)\}$. We will examine $C_G(i)^{(x_1, x_2)}$ as the intersection of $C_G(i)^{(x_1)}$ and $C_G(i)^{(x_2)}$ similarly to the previous proof. Let us see the $\mathcal{U}_1 - \mathcal{U}_1$ table for $C_G(i)$.

	1	H_2	H_4	H_{2q+2}	$C_P(i)$	$C_G(i)$
1	1	1	1	1	1	1
H_2	1	$1, H_2$	$1, H_2$	$1, H_2$	1	H_2
H_4	1	$1, H_2$	$1, H_2, H_4$	$1, H_2, H_4$	1	H_4
H_{2q+2}	1	$1, H_2$	$1, H_2, H_4$	$\langle i \rangle, V, S, H_{2q+2}$	1	H_{2q+2}
$C_P(i)$	1	1	1	1	$1, C_P(i)$	$C_P(i)$
$C_G(i)$	1	H_2	H_4	H_{2q+2}	$C_P(i)$	$C_G(i)$

The only relevant case is, when $C_G(i)^{(x_1)}, C_G(i)^{(x_2)} \simeq C_2^2 \times D_{\frac{q+1}{2}}$. Then $[i, i^{x_1}] = [i, i^{x_2}] = 1$. Let us define the subgroups $V_1 := \langle i, i^{x_1} \rangle$ and $V_2 := \langle i, i^{x_2} \rangle$. Then we have that $C_G(V_l) = C_G(i)^{(x_l)}$ for $l = 1, 2$. Furthermore, we introduce the following notation: $C_G(i) \geq C_G(V_l) = V_l \times (M_l \rtimes \langle i_l \rangle) \simeq V_l \times D_{\frac{q+1}{2}}$ for $l = 1, 2$, i.e. M_l denotes the cyclic subgroups of order $\frac{q+1}{4}$ in $C_G(V_l)$. Since $C_G(i) \simeq \langle i \rangle \times L$, where $L \simeq PSL(2, q)$, both M_1 and M_2 are subgroups of the second component in this direct product. According to Theorem 2.1 (e), these two subgroups either coincide, or intersect trivially.

H_{2q+2} : If $M_1 = M_2$ then since $C_G(M_l) = V_l \times M_l$, we have that $V_1 = V_2$ and then $C_G(i)^{(x_1, x_2)} = C_G(V_1) \simeq H_{2q+2}$.

If they intersect trivially, then $V_1 \neq V_2$ and the intersection $C_G(V_1) \cap C_G(V_2) = C_G(i)^{(x_1, x_2)}$ is a 2-subgroup. We will show that for a suitable choice of x_1, x_2 the intersection can be S, V or $\langle i \rangle$.

S : Let us choose the elements x_1, x_2 in such a way that $S := \langle i, i^{x_1}, i^{x_2} \rangle \in Syl_2(G)$, this is possible since the involutions of G are conjugate. Let V_1, V_2 be as above. Then $V_1 \neq V_2$, thus $C_G(V_1) \cap C_G(V_2)$ is a 2-group. Since it contains S , it must be S .

V : Let $V = \langle i, j \rangle$, where i and j are involutions in G with $[i, j] = 1$. We want to show that there are two involutions i^{x_1}, i^{x_2} such that $V = C_G(i)^{(x_1, x_2)}$. We know that $C_G(i) \cap C_G(j) = \langle i, j \rangle \times D_{\frac{q+1}{2}}$. Let us choose two different involutions k_1, k_2 , from $D_{\frac{q+1}{2}}$. Let $V_1 := \langle i, k_1 \rangle, V_2 := \langle i, k_2 \rangle$. Then $V_1 \neq V_2$ thus $C_G(V_1) \cap C_G(V_2)$ is a 2-group. However, each Sylow 2-subgroup of $C_G(V_1)$ contains V_1 and each Sylow 2-subgroup of $C_G(V_2)$ contains V_2 , thus the intersection cannot be a Sylow 2-subgroup, since $[k_1, k_2] \neq 1$. So the intersection is V . Since the involutions are all conjugate in G , thus $k_1 = i^{x_1}, k_2 = i^{x_2}$, for suitable elements x_1, x_2 . Hence, $V = C_G(i)^{(x_1, x_2)}$.

$\langle i \rangle$: Now we construct V_1, V_2 in such a way that $C_G(V_1) \cap C_G(V_2) = \langle i \rangle$. Let $[i, i^{x_1}] = 1$. Then $C_G(i) \cap C_G(i^{x_1}) = \langle i \rangle \times (L \cap C_G(i^{x_1})) = \langle i \rangle \times D$, where $D \leq L \simeq PSL(2, q)$ is a dihedral subgroup of order $q + 1$. However by Lemma 2.2 in [3], for each dihedral subgroup D of $L \simeq PSL(2, q)$ there is an element $g \in L$ such that $D^g \cap D = \{1\}$. Thus if we choose g as above then $g \in C_G(i)$ and $C_G(i) \cap C_G(i^{x_1 g}) = \langle i \rangle \times D_{q+1}^g$ and hence $C_G(i)^{(x_1, x_1 g)} = \langle i \rangle$. Since H_{2q+2} always contains i , no other intersection is possible. \square

Theorem 6.4. *We have that $d_c(C_G(i), G) = 6$ and $d(C_G(i), G) = 3$.*

Proof. Now we set up a $(\mathcal{U}_2 \setminus \mathcal{U}_1) - \mathcal{U}_1$ table to study the types of subgroups $Z \in \mathcal{U}_3(C_G(i))$. We know that Z is the intersection of X and Y , where $X \in \mathcal{U}_1(C_G(i))$ and $Y \in \mathcal{U}_2(C_G(i))$. Obviously the intersection of two elements of $\mathcal{U}_1(C_G(i))$ is in $\mathcal{U}_2(C_G(i))$, therefore we need to examine only the case when Y is a new content of $\mathcal{U}_2(C_G(i))$. We will use the same notation for the subgroup types of $\mathcal{U}_1(C_G(i))$ as in the proof of Proposition 6.3. Denote the subgroup types in $\mathcal{U}_2(C_G(i)) \setminus \mathcal{U}_1(C_G(i))$ by $\langle i \rangle, V$ and S , respectively, where V is a Klein subgroup containing i and S is a Sylow 2-subgroup containing i .

	1	H_2	H_4	H_{2q+2}	$C_P(i)$	$C_G(i)$
$\langle i \rangle$	1	1	1	$\langle i \rangle$	1	$\langle i \rangle$
V	1	$1, H_2$	$1, H_2$	$\langle i \rangle, V$	1	V
S	1	$1, H_2$	$1, H_2, H_4$	$\langle i \rangle, V, S$	1	S

Now we have, that $\mathcal{U}_2(C_G(i)) = \mathcal{U}_3(C_G(i))$ and thus, $d_c(C_G(i), G)$ is 5 or 6 by condition (1) in Definition 1.1. Using this, we will show, that the combinatorial depth is 6.

Let $x_1 \in C_G(i)$ and let x_2, x_3 be elements in G such that $C_G(i)^{(x_2)} = C_G(\langle i, k_1 \rangle)$, $C_G(i)^{(x_3)} = C_G(\langle i, k_2 \rangle)$ where i, k_1, k_2 are involutions as in Proposition 6.3, at the construction of V . Then $[i, k_1] = [i, k_2] = 1 \neq [k_1, k_2]$. We also have that $i \in C_G(i)^{(x_1, x_2, x_3)} = V$. Let us suppose that we have elements $y_1, y_2 \in G$ such that $C_G(i)^{(y_1, y_2)} = C_G(i)^{(x_1, x_2, x_3)}$ and $x_1 \in C_G(i)$ acts in the same way as y_1 . Then $y_1 \in C_G(i)$. Therefore $V = C_G(i)^{(y_1, y_2)} = C_G(i)^{(y_2)} \in \mathcal{U}_1(C_G(i))$, which is a contradiction. Thus, $d_c(C_G(i), G) = 6$. Finally, by Theorem 1.3 we get that $d(C_G(i), G) = 3$. \square

7. THE DEPTH OF G_0 , IF $G_0 \not\cong R(3)$

In this section we consider a maximal subgroup $G_0 \simeq R(q_0)$ of G , where $q_0 = 3^{2n_0+1} = 3m_0^2$ for $n_0 \geq 1$. We will study which subgroups of G can occur in the form $G_0^{(g)} \neq G_0$, by considering the maximal subgroups of G_0 that can contain them. Let H be a maximal subgroup of G_0 . Let us introduce the notation $U_H = \{G_0^{(g)} \mid g \in G \setminus G_0 \text{ and } G_0^{(g)} \leq H\}$. Then we have that $\mathcal{U}_1(G_0) = G_0 \cup \bigcup_{H \in \text{Max}(G_0)} U_H$.

The following Lemma shows that in several cases $G_0^{(g)}$ is already in $\mathcal{U}_1(H)$, for some $H < G_0$.

Lemma 7.1. *a) Let $H_0, T_0 \leq G$ and $g \in G$. If $H_0 \leq T_0^{(g)}$, and the subgroups of T_0 of order $|H_0|$ are conjugate in T_0 , then $g \in T_0 N_G(H_0)$.*
b) Let $T_0, H \leq G$ and $g \in G$. If there is an element $t \in T_0$ such that $t^{-1}g \in N_G(H)$ and $T_0^{(g)} \leq H$, then $T_0^{(g)} = (T_0 \cap H)^{(t^{-1}g)}$.
c) Let $H \leq G$. If there exists an element $r \in G_0$ such that for a fixed $g \in G$, $r^{-1}g \in N_G(H)$, then $G_0^{(g)} \cap N_G(H) = (N_{G_0}(H))^{(r^{-1}g)}$.
d) Let $H_0 \leq G_0$ such that $N_{G_0}(H_0)$ is the only maximal subgroup in G_0 containing H_0 . If there exists an element $r \in G_0$ such that for a fixed $g \in G \setminus G_0$, $r^{-1}g \in N_G(H_0)$, then $G_0^{(g)} = N_{G_0}(H_0)^{(r^{-1}g)}$.

Proof. a) Since $H_0^{g^{-1}} \leq T_0$, there is an element $t \in T_0$ such that $H_0^{g^{-1}t} = H_0$ and $g^{-1}t \in N_G(H_0)$.

b) It can be easily checked.

c) $G_0^{(g)} \cap N_G(H) = G_0^{(r^{-1}g)} \cap N_G(H) = N_{G_0}(H) \cap N_{G_0^{r^{-1}g}}(H) = N_{G_0}(H) \cap (N_{G_0}(H))^{r^{-1}g}$

d) It is easy to see that $G_0^{(g)} \geq (N_{G_0}(H_0))^{(r^{-1}g)}$, thus $H_0 \leq G_0^{(g)}$. By assumption, $G_0^{(g)} \neq G_0$. Since there is only one maximal subgroup containing H_0 in G_0 and $G_0^{r^{-1}g}$, respectively, we have that $G_0^{(g)} \leq N_{G_0}(H_0) \cap N_{G_0^{r^{-1}g}}(H_0) = (N_{G_0}(H_0))^{(r^{-1}g)}$. \square

The following Lemma shows that if we can find elements of certain type in $G_0^{(g)}$, then their centralizer in G_0 also lies in $G_0^{(g)}$.

Lemma 7.2. *a) If a non-trivial element $x \in G$, whose order divides $\frac{q_0+1}{4}$, is contained in $G_0^{(g)}$, then $C_{G_0}(x) (\simeq C_2^2 \times C_{\frac{q_0+1}{4}}) \leq G_0^{(g)}$.*

- b) If a non-trivial element $x \in G$, whose order divides $\frac{q_0-1}{2}$, is contained in $G_0^{(g)}$, then $C_{G_0}(x) \simeq C_{q_0-1} \leq G_0^{(g)}$.
- c) If a non-trivial element $x \in G$, whose order divides either $q_0 + 3m_0 + 1$ or $q_0 - 3m_0 + 1$, is contained in $G_0^{(g)}$, then $C_{G_0}(x) \leq G_0^{(g)}$. (Here $C_{G_0}(x)$ is isomorphic to $C_{q_0+3m_0+1}$ or $C_{q_0-3m_0+1}$, respectively.)
- d) If a Klein subgroup V is contained in $G_0^{(g)}$, then the unique subgroup M_0 of order $\frac{q_0+1}{4}$ of $C_G(V) = V \times (M \rtimes C_2)$ is a subgroup of $G_0^{(g)}$, thus $C_{G_0}(M_0) = V \times M_0 \leq G_0^{(g)}$.

Proof. a) By Lemma 2.3 we know that $C_{G_0}(x)$, $C_{G_0^g}(x) \simeq C_2^2 \times C_{\frac{q_0+1}{4}}$ and $C_G(x) \simeq C_2^2 \times C_{\frac{q+1}{4}}$.

Since $C_G(x)$ contains only one subgroup of order q_0+1 , we have that $C_{G_0}(x) = C_{G_0^g}(x) \leq G_0^{(g)}$.

- b) By Theorem 2.3 we know, that both $C_{G_0}(x)$ and $C_{G_0^g}(x)$ are isomorphic to C_{q_0-1} and $C_G(x) \simeq C_{q-1}$. Since $C_G(x)$ contains only one subgroup of order q_0-1 , we have that $C_{G_0}(x) = C_{G_0^g}(x) \leq G_0^{(g)}$.
- c) We consider the case, when the order of x divides $q_0 + 3m_0 + 1$, the other case is similar. By Theorem 2.1 (d) we know that $C_{G_0}(x)$, $C_{G_0^g}(x) \simeq C_{q_0+3m_0+1}$ and $C_G(x) = C_2^2 \times C_{\frac{q+1}{4}}$ or $C_{3q \pm 3m+1}$ depending on, whether $\frac{q+1}{4}$ or $q \pm 3m + 1$ is divisible by $q_0 + 3m_0 + 1$. In each case there is only one subgroup of order $q_0 + 3m_0 + 1$ in $C_G(x)$ and hence $C_{G_0}(x) = C_{G_0^g}(x) \leq G_0^{(g)}$.
- d) Let us suppose that $G_0^{(g)}$ contains a Klein subgroup V . Observe that the subgroup M_0 of order $\frac{q_0+1}{4}$ of $C_G(V) = V \times (M \rtimes C_2)$ is unique. However, $C_{G_0}(V)$ and $C_{G_0^g}(V)$ are both isomorphic to $V \times (C_{\frac{q_0+1}{4}} \rtimes C_2)$ and they are contained in $C_G(V)$. Thus $M_0 \leq G_0^{(g)}$, and hence $C_{G_0}(M_0) = V \times M_0 \leq G_0^{(g)}$. □

Now we determine $U_{N_{G_0}(P_0)}$ for a fixed Sylow 3-subgroup P_0 of G_0 .

Lemma 7.3. *Let P be a Sylow 3-subgroup of the Ree group G and let W be an arbitrary complement of P in $N_G(P)$. Let $W_{2'}$ be the $2'$ part of W . Then:*

- a) *Every nontrivial element of W acts regularly on $Z(P) \setminus \{1\}$ via the conjugation action.*
- b) *The conjugation action of every nontrivial element $w \in W_{2'}$ on $P'/Z(P)$ has only one fixed point, namely $Z(P)$. Furthermore, the conjugation action of $W_{2'}$ has 3 orbits on $P'/Z(P)$, where $O_1 = Z(P)$ and $O_2^{-1} = O_3$.*
- c) *The conjugation action of every nontrivial element $w \in W_{2'}$ on P/P' has only one fixed point, P' .*

Proof. Obviously $Z(P)$ and P' are W -invariant.

- a) By Theorem 2.1 (b), for every nontrivial element $w \in W_{2'}$, we have that $C_{Z(P)}(w) = \{1\}$. The same holds for order 2 elements of W . Hence, $C_{Z(P)}(w) = 1$, for every nontrivial element $w \in W$. Since $|W| = |Z(P) \setminus \{1\}|$, the action is regular.
- b) By Theorem 2.1 (b), for every nontrivial element $w \in W_{2'}$, we have that $C_{P'}(\langle w \rangle) = \{1\}$. Using [8, Theorem 3.15, Ch. 5, p. 187], we have that $C_{P'/Z(P)}(\langle w \rangle) = \{\bar{1}\}$. Obviously, an element and its inverse cannot belong to the same $W_{2'}$ -orbit. Since $|W_{2'}| = \frac{|P'/Z(P)|-1}{2}$, we are done.
- c) We use again [8, Theorem 3.15, Ch. 5, p.187] to deduce that $C_{P/P'}(\langle w \rangle) = \{\bar{1}\}$. □

Proposition 7.4. *Let G be a Ree group, G_0 a Ree-subgroup of G and P a Sylow 3-subgroup of G . Then $P \cap G_0$ is either trivial or a Sylow 3-subgroup of G_0 .*

Proof. Assume, that the intersection of G_0 and P is not trivial. In this case there is a Sylow 3-subgroup P_0 of G_0 , containing the intersection. Since the Sylow 3-subgroups in G_0 are TI, P_0 is unique. If P would not contain P_0 , then there would be another Sylow 3-subgroup R of G , which would contain it. However, in this case P and R would have common elements, and this would contradict the fact, that the Sylow 3-subgroups of G are TI. Thus, we have that $G_0 \cap P$ contains P_0 and hence they are equal. □

Notation 7.5. *We fix the following notation. Let G_0 be a fixed maximal subgroup of G isomorphic to $R(q_0)$. Let P_0 be a Sylow 3-subgroup of G_0 . Let $P \in \text{Syl}_3(G)$ containing P_0 . Let us suppose that in the representation of P of Theorem 2.1 (h), the elements of P_0 correspond to triples $\{(a, b, c) | a, b, c \in GF(q_0)\}$. Let W_0 be a complement of P_0 in $N_{G_0}(P_0)$ let W be the complement of P in $N_G(P)$ containing W_0 . Let i be the unique involution in W_0 .*

Lemma 7.6. *We use Notation 7.5.*

- a) Let $x \in G$ and let j be an involution in P_0W_0 . Then $j \in (P_0W_0)^{(x)}$ if and only if $x \in P_0C_G(j)$.
b) Let $W_1 \leq P_0W_0$ of order $q_0 - 1$. Then $W_1 \leq (P_0W_0)^{(x)}$ if and only if $x \in P_0N_G(W_1)$.

Proof. a) \Rightarrow By assumption $j, j^{x^{-1}} \in P_0W_0$. Since the involutions are conjugate in P_0W_0 , we have that there exists an element $p_0 \in P_0$ such that $j^{p_0} = j^{x^{-1}}$, thus $p_0x \in C_G(j)$ and $x \in P_0C_G(j)$.
 \Leftarrow By assumption there is an element $p_0 \in P_0$ such that $p_0x \in C_G(j)$, thus $(P_0W_0)^{(x)} = (P_0W_0)^{(p_0x)}$, which contains j .

b) The proof is similar to that of a). \square

Proposition 7.7. *Using Notation 7.5 we have that*

$$U_{N_{G_0}(P_0)} = \{Z(P_0), P'_0, P'_0\langle i^{p_0} \rangle \mid p_0 \in P_0\} \cup U \cup V,$$

where $V \subseteq \{Z(P_0)\langle h \rangle, P_0, P_0\langle i \rangle, P_0W_0 \mid h \in P_0 \setminus P'_0, p_0 \in P_0\}$ and $U \subseteq \{1, \langle i^{p_0} \rangle, W_0^{p_0} \mid p_0 \in P_0\}$.

Proof. It directly follows from: \square

Lemma 7.8. *Let $g \in G$ be an element such that $G_0^{(g)} \neq G_0$. Then*

- (A) If $G_0^{(g)}$ contains a 3-element $p \in P_0$, then there exists an element $r \in G_0$ such that $r^{-1}g \in N_G(P)$. Moreover, this element r can be chosen so that $r^{-1}g = p_1 \in P \setminus P_0 = (Z(P) \setminus Z(P_0)) \cup (P' \setminus P'_0Z(P)) \cup (P \setminus P_0P')$ and $G_0^{(g)} = (P_0W_0)^{(r^{-1}g)} = (P_0W_0)^{(p_1)}$. In particular, $Z(P_0) \leq G_0^{(g)}$ and $G_0^{(g)} \leq N_{G_0}(P_0)$.
I. If $r^{-1}g \in Z(P) \setminus Z(P_0)$ then $G_0^{(g)} = P_0, P_0\langle i \rangle$ or P_0W_0 .
II. If $r^{-1}g \in P' \setminus P'_0Z(P)$ then $G_0^{(g)}$ is isomorphic to P'_0 or $P'_0\langle i \rangle$. Moreover, there exists an element $p_1 \in P' \setminus P'_0Z(P)$ such that $G_0^{(p_1)} = P'_0$ and for every involution $j \in P_0W_0$ there exists an element $p_2 \in P' \setminus P'_0Z(P)$ such that $G_0^{(p_2)} = P'_0\langle j \rangle$. The case P'_0W_0 does not occur.
III. If $r^{-1}g \in P \setminus P_0P'$ then $G_0^{(g)}$ is $Z(P_0)$ or $Z(P_0)\langle h \rangle$, where h is an element of order 9 in P_0 . Moreover, there exists an element $p_1 \in P \setminus P_0P'$ such that $G_0^{(p_1)} = Z(P_0)$.
(B) If $G_0^{(g)} \leq N_{G_0}(P_0)$ and $G_0^{(g)}$ does not contain any nontrivial 3-elements from P_0 , then $G_0^{(g)}$ is isomorphic to $1, C_2$ or C_{q_0-1} .

Proof. Case (A) Let $p \in G_0^{(g)}$ be a nontrivial 3-element belonging to P_0 . Let $S_0 \in \text{Syl}_3(G_0)$ containing $p^{g^{-1}}$. Then there exists an element $r \in G_0$ such that $S_0^r = P_0$. Since $p, p^{g^{-1}r} \in P_0 \leq P$, and P is TI, we have that $r^{-1}g \in N_G(P)$. Thus $g \in G_0N_G(P) \setminus G_0$. Then, since $G_0WP = G_0W_0P \cup G_0(W \setminus W_0)P = G_0(P \setminus P_0) \cup G_0 \cup G_0(W \setminus W_0)P$, we have that $G_0N_G(P) \setminus G_0 = G_0(P \setminus P_0) \cup G_0(W \setminus W_0)P$. We will show that if $g \in G_0(W \setminus W_0)P$ then $G_0^{(g)}$ does not contain any non-trivial elements from P_0 . Thus in Case (A) $g \in G_0(P \setminus P_0)$ must hold.

Let $g = rw_1p_1$, where $r \in G_0$, $w_1 \in W \setminus W_0$ and $p_1 \in P$. We may suppose that $w_1 \in W_{2'}$, otherwise, since $g = (ri)(iw_1)p_1$, we choose $ri \in G_0$ instead of r and $iw_1 \in W \setminus W_0$ instead of w_1 . Using the fact that $N_{G_0}(P_0) \leq N_{G_0}(P)$ and Lemma 7.1 c), with $H := P$ and $r := r$, we have that $G_0^{(g)} \cap N_G(P) = N_{G_0}(P)^{(w_1p_1)} = N_{G_0}(P_0)^{(w_1p_1)} = (P_0W_0)^{(w_1p_1)}$.

Let $1 \neq z \in Z(P_0) \cap (P_0W_0)^{w_1p_1}$. Then $z \in (P_0W_0)^{w_1}$ and $z^{w_1^{-1}} \in P_0 \cap Z(P) = Z(P_0)$. Let $z_0 = z^{w_1^{-1}}$. Then there exists an element $w_0 \in W_0$ such that $z^{w_0} = z_0$. Hence $z^{w_0w_1} = z$. Using Lemma 7.3 a) we have a contradiction. Hence $(P_0W_0)^{(w_1p_1)}$ does not contain any nontrivial elements from $Z(P_0)$.

Hence $P_0W_0^{(w_1p_1)}$ neither contains any elements of order 9 from P_0 . We show that $(P_0W_0)^{(w_1p_1)}$ does not contain any other 3-elements. Let h be an element of order 3 in $(P_0W_0)^{(w_1p_1)}$. Then $h \in P'_0 \setminus Z(P_0)$. Using the representation of the Sylow 3-subgroups $P_0 \in \text{Syl}_3(G_0)$ and $P \in \text{Syl}_3(G)$, we have that $h = (0, y_0, z_0)$ and $p_1^{-1} = (a, b, c)$, where $y_0 \neq 0$, $y_0, z_0 \in GF(q_0)$ and $a, b, c \in GF(q)$. Denote w_1^{-1} by w . Then by Theorem 2.1 (h) and (by Lemma 2.6) we have that $h^{p_1^{-1}w} = (0, y_0, z_0)^w(0, 0, 2y_0a)^w \in P_0 \cap P' = P'_0$. Denote the element $(0, 0, 2y_0a)^w$ by ζ . Since $(0, y_0, z_0)^w\zeta \in P'_0 \setminus Z(P_0)$, applying Lemma 7.3 b) for P'_0 we have that there are elements $w_0 \in (W_0)_{2'}$ and $\zeta_0 \in Z(P_0)$ such that $(0, y_0, z_0)^w\zeta = (0, y_0, z_0)^{w_0}\zeta_0$ or $(0, y_0, z_0)^w\zeta = ((0, y_0, z_0)^{-1})^{w_0}\zeta_0$. Using Lemma 7.3 b) again for P' , we have in both cases a contradiction. In the first case $ww_0^{-1} \in W_{2'}$ fixes $(0, y_0, z_0)Z(P)$, and in the second case ww_0^{-1} takes $(0, y_0, z_0)Z(P)$ to its inverse. Thus $G_0^{(g)} \cap P_0 = \{1\}$, which is a contradiction. Thus $g \notin G_0(W \setminus W_0)P$.

Hence if $G_0^{(g)}$ contains a nontrivial 3-element $p \in P_0$, then $g \in G_0(P \setminus P_0)$ i. e. $g = rp_1$ where $r \in G_0$ and $p_1 \in P \setminus P_0$. Thus by Lemma 7.1 d) with $Z(P_0)$ as H_0 , we have that $G_0^{(g)} = (N_{G_0}(Z(P_0)))^{(p_1)} = (P_0W_0)^{(p_1)}$.

Obviously, $P \setminus P_0$ is a disjoint union of $Z(P) \setminus Z(P_0)$, $P' \setminus P'_0 Z(P)$ and $P \setminus P_0 P'$. Hence $p_1 = r^{-1}g$ belongs to one of them.

- I. If $r^{-1}g = p_1 \in Z(P) \setminus Z(P_0)$ then as $[P_0, p_1] = 1$, $G_0^{(g)} = (P_0 W_0)^{(p_1)} \geq P_0$. Using Lemma 7.2 b) we have that $G_0^{(g)}$ is $P_0, P_0 \langle i \rangle$ or $P_0 W_0$.
- II. If $r^{-1}g = p_1 \in P' \setminus P'_0 Z(P)$, then $G_0^{(g)} = (P_0 W_0)^{(p_1)} \geq P'_0$, since P' is elementary abelian. If $G_0^{(g)} = (P_0 W_0)^{(p_1)}$ would contain an element h of order 9, then $h^{p_1^{-1}} \in P_0$ would hold. Using the representation of elements of the Sylow 3-subgroup of G in Theorem 2.1 (h) and Lemma 2.6, we have that if $p_1^{-1} = (0, b, c)$ and $h = (x_0, y_0, z_0)$ with $b \notin GF(q_0)$, $x_0 \neq 0$ and $x_0, y_0, z_0 \in GF(q_0)$, then $h^{p_1^{-1}} = (x_0, y_0, z_0 - 2x_0 b) \in P_0$, which is a contradiction. Hence, again by Lemma 7.2 b), $G_0^{(g)}$ can only be isomorphic to $P'_0, P'_0 \rtimes C_2$ or $P'_0 \rtimes C_{q_0-1}$.
Now we prove that $G_0^{(g)} = (P_0 W_0)^{(p_1)}$ is not isomorphic to $P'_0 \rtimes C_{q_0-1}$. Assume that $W_1 = W_0^{p_0} \leq (P_0 W_0)^{(p_1)}$ for some $p_0 \in P_0$. By Lemma 7.6 b) $W_1 \leq (P_0 W_0)^{(p_1)}$ if and only if $p_1 \in P_0 N_G(W_1)$. Thus we have that $p_1 \in P_0 N_G(W_1)$, in particular, $p_1 \in P_0 N_P(W_1)$. However by Theorem 2.1 (b) we have that $N_G(W_1) = W_1 \rtimes C_2$, hence $N_P(W_1) = 1$. Thus $p_1 \in P_0$, which contradicts our assumption, that $p_1 \in P' \setminus P'_0 Z(P)$.

Now we show, that for every involution $j \in P_0 W_0$ if we take $p_1 \in C_{P \setminus P_0}(j)$, which is nontrivial, then $G_0^{(p_1)} = P'_0 \langle j \rangle$. By Theorem 2.1 (b) we have that $p_1 \in P' \setminus Z(P)$ and $p_1 \in P' \setminus P'_0$. We show that $p_1 \in P' \setminus P'_0 Z(P)$. For this, consider the representation of the Sylow 3-subgroup of G in Theorem 2.1 (h). By Theorem 2.1 (b) we have that $C_P(j) \cap Z(P) = \{1\}$. We prove that for every $b \in GF(q)$ there is at most one $c \in GF(q)$ such that $(0, b, c) \in C_P(j)$. Let us suppose that $(0, b, c_1), (0, b, c_2) \in C_P(j)$. Then their quotient, $(0, 0, c_1 - c_2) \in C_P(j) \cap Z(P) = \{1\}$. Hence $c_1 = c_2$. Since $|C_P(j)| = q$, for every $b \in GF(q)$ there is exactly one $c \in GF(q)$ with $(0, b, c) \in C_P(j)$. Similar statement holds for $C_{P_0}(j)$. Hence, if $b \in GF(q_0)$, then the unique c must be in $GF(q_0)$ and so $C_{P'_0 Z(P)}(j) = C_{P'_0}(j)$. Thus, $p_1 \in (P' \setminus P'_0) \cap C_P(j)$ implies that $p_1 \in P' \setminus P'_0 Z(P)$. Since $p_1 \in N_G(P)$, we have seen at the beginning of the proof that $G_0^{(p_1)} = (P_0 W_0)^{(p_1)}$. Using that $p_1 \in P' \setminus P'_0 Z(P)$, we proved already that $(P_0 W_0)^{(p_1)}$ is isomorphic to P'_0 or $P'_0 \langle i \rangle$. By Lemma 7.6 we have that $j \in (P_0 W_0)^{(p_1)}$. Hence $G_0^{(p_1)} = P'_0 \langle j \rangle$.

Finally we prove, that there is an element $p_1 \in P' \setminus P'_0 Z(P)$ such that $(P_0 W_0)^{(p_1)}$ does not contain any involutions. Let $j \in (P_0 W_0)^{(p_1)}$ be a fixed involution. Using Lemma 7.6 for $x = p_1 \in P' \setminus P'_0 Z(P)$, by Theorem 2.1 (b) and (h) we have that $|\{p_1 \in P' \setminus P'_0 Z(P) \mid j \in (P_0 W_0)^{(p_1)}\}| = |(P' \setminus P'_0 Z(P)) \cap (P_0 C_P(j))| \leq |P'_0 C_P(j) \setminus P'_0| = (q - q_0)q_0$.

If for some element $p_1 \in P' \setminus P'_0 Z(P)$, the subgroup $(P_0 W_0)^{(p_1)}$ contains an involution j , then $(P_0 W_0)^{(p_1)} = P'_0 \langle j \rangle$. Since the involutions are conjugate in $P_0 W_0$ by elements of P_0 , for every involution $j \in P_0 W_0$ the same number of $p_1 \in P' \setminus P'_0 Z(P)$ occurs such that $(P_0 W_0)^{(p_1)}$ contains j . Since by Lemma 2.4 we have that $C_P(p_1) = P'$, the cosets of P'_0 in P_0 move p_1 to q_0 different places. Thus, from the number of elements p_1 belonging to the involution j , one can get an upper bound on the number of elements p_1 belonging to any involution in $P_0 W_0$, by multiplying with q_0 . Hence, we have that $|\{p_1 \in P' \setminus P'_0 Z(P) \mid (P_0 W_0)^{(p_1)} \text{ contains involutions}\}| \leq (q - q_0)q_0^2$. Since $|P' \setminus P'_0 Z(P)| = q(q - q_0)$ is bigger than $q_0^2(q - q_0)$, there is an element $p_1 \in P' \setminus P'_0 Z(P)$ such that $(P_0 W_0)^{(p_1)}$ does not contain involutions. Then, by the beginning of the proof of II., we have that $(P_0 W_0)^{(p_1)} = P'_0$ and hence $G_0^{(p_1)} = (P_0 W_0)^{(p_1)} = P'_0$.

- III. If $r^{-1}g = p_1 \in P \setminus P_0 P'$ then $G_0^{(g)} = (P_0 W_0)^{(p_1)} \geq Z(P_0)$. We show that $(P_0 W_0)^{(p_1)}$ cannot contain noncentral elements of order 3 in P_0 . As before, we use the representation in Theorem 2.1 (h) of P and P_0 . Let us suppose that $h \in (P_0 W_0)^{(p_1)}$ is a noncentral element of order 3 in P_0 . Let $p_1^{-1} = (a, b, c)$ and $h = (0, y_0, z_0) \in P_0$. Then $a \notin GF(q_0)$, otherwise $(a, b, c) = (a, 0, 0)(0, b, c + ab) \in P_0 P'$, which is not the case. On the other hand, $y_0 \neq 0$, since otherwise $h \in Z(P_0)$. Then by Lemma 2.6, $h^{p_1^{-1}} = (0, y_0, z_0 + 2y_0 a) \in P_0$, which is a contradiction.

Now we prove that $G_0^{(g)} = (P_0 W_0)^{(p_1)}$ does not contain any elements outside P_0 . As before, using Lemma 7.2 b), it is enough to show that $(P_0 W_0)^{(p_1)}$ does not contain involutions. Suppose this is not true and let j be an involution in $(P_0 W_0)^{(p_1)}$. By Lemma 7.6 we have that $p_1 \in P_0 C_P(j)$ and by Theorem 2.1 (b) we have that $p_1 \in P_0 P'$, which contradicts our assumption.

Now we show that if there is an element $h \in P_0$ of order 9 in $(P_0W_0)^{(p_1)}$ then $(P_0W_0)^{(p_1)} = Z(P_0)\langle h \rangle$. We use the representation of the Sylow 3-subgroups P_0 and P , as before. Let $p_1^{-1} = (a, b, c)$ and $h = (x_0, y_0, z_0)$. Clearly $a \notin GF(q_0)$, $x_0, y_0, z_0 \in GF(q_0)$ and $x_0 \neq 0$. Then by Lemma 2.6, we have that

$h^{p_1^{-1}} = (x_0, y_0 + x_0(a\sigma) - a(x_0\sigma), z_0 - 2x_0b + 2y_0a - ax_0(x_0\sigma) + ax_0(a\sigma)) \in P_0$. Consider the solutions $x \in GF(q_0) \setminus \{0\}$ of the relation $x(a\sigma) - a(x\sigma) \in GF(q_0)$. Applying σ to this relation, as σ is also an automorphism of $GF(q_0)$, we have that $(x\sigma)a^3 - (a\sigma)x^3 \in GF(q_0)$. Multiplying by x^2 on the left hand side of the original relation, adding the left hand side of the second to it and dividing by $(x\sigma)$, we have that $a^3 - ax^2 \in GF(q_0)$. Recall that x_0 and $-x_0$ both satisfy this relation. If some $x \in GF(q_0) \setminus \{0\}$ also satisfies it, then $a^3 - a(x_0)^2 - (a^3 - ax^2) = a(x_0 - x)(x_0 + x) \in GF(q_0)$. Since $a \notin GF(q_0)$, we have that either $x = x_0$ or $x = -x_0$. Let $h' = (x', y', z') \in (P_0W_0)^{(p_1)}$ be another element. As we have seen, $h' \in P_0$. If $h' \notin Z(P_0)$, then it is of order 9. Then $h'^{p_1^{-1}} = (x', y' + x'(a\sigma) - a(x'\sigma), z' - 2x'b + 2y'a - ax'(x'\sigma) + ax'(a\sigma)) \in P_0$. Then x' also satisfies $a^3 - ax' \in GF(q_0)$, thus $x' = x_0$ or $x' = -x_0$ holds.

If $x' = x_0$ then by subtracting the 3-rd component of $h'^{p_1^{-1}}$ from that of $h^{p_1^{-1}}$ we have that $2a(y_0 - y') \in GF(q_0)$. Since $a \notin GF(q_0)$ we have that $y_0 - y' = 0$, hence $h' \in hZ(P_0)$.

If $x' = -x_0$ then by adding the third component of $h'^{p_1^{-1}}$ to that of $h^{p_1^{-1}}$, we have that $2a(y_0 + y' - x_0(x_0\sigma)) \in GF(q_0)$. Since $a \notin GF(q_0)$, we have that $y' = -y_0 + x_0(x_0\sigma)$, thus by Lemma 2.6, we have that $h' \in h^{-1}Z(P_0)$ and hence $(P_0W_0)^{(p_1)}$ is equal to $Z(P_0)\langle h \rangle$.

Thus we have seen that $p_1 \in P \setminus P_0P'$ implies that $(P_0W_0)^{(p_1)}$ can be either $Z(P_0)\langle h \rangle$ for some element $h \in P_0$ of order 9, or $(P_0W_0)^{(p_1)} = Z(P_0)$.

We show that there is an element $p_1 \in P \setminus P_0P'$ such that $G_0^{(p_1)} = (P_0W_0)^{(p_1)} = Z(P_0)$.

Let $p_1^{-1} = (a, b, c) \in P \setminus P_0P'$. Let $h = (x_0, y_0, z_0) \in P_0$ be an element such that $x_0 \neq 0$, i.e. it is of order 9, and let $h \in (P_0W_0)^{(p_1)}$. Then $h^{p_1^{-1}} \in P_0$ hence x_0, y_0, z_0 satisfies

$$x_0(a\sigma) - a(x_0\sigma) \in GF(q_0) \quad (1)$$

and

$$-2x_0b + 2y_0a - ax_0(x_0\sigma) + ax_0(a\sigma) \in GF(q_0) \quad (2)$$

Let $(x_0, y_0, z_0) \in P_0$ be fixed such that $x_0 \neq 0$. Then

$$|\{p_1 \in P \setminus P_0P' \mid (x_0, y_0, z_0) \in (P_0W_0)^{(p_1)}\}| \leq |\{(a, b, c) \in GF(q)^3 \mid a \notin GF(q_0), \text{ and } b \text{ satisfies (2)}\}| \leq (q - q_0)q_0q.$$

If $p_1 \in P \setminus P_0P'$ and $(P_0W_0)^{(p_1)}$ contains an element h of order 9, then $(P_0W_0)^{(p_1)} = Z(P_0)\langle h \rangle$. It contains exactly $2q_0$ elements of order 9, since $h^3 \in Z(P_0)$.

So there are $\frac{q_0^3 - q_0^2}{2q_0}$ elements of order 9 in P_0 giving different subgroups in P_0 isomorphic to $Z(P_0)\langle h \rangle$. Thus

$$|\{p_1 \in P \setminus P_0P' \mid (P_0W_0)^{(p_1)} \text{ contains elements of order 9}\}| \leq \frac{q_0^3 - q_0^2}{2q_0} q_0q(q - q_0).$$

Since $|P \setminus P_0P'| = q^2(q - q_0)$ is bigger than $\frac{1}{2}q(q_0^3 - q_0^2)(q - q_0)$, we have that there is an element p_1 such that $G_0^{(p_1)} = (P_0W_0)^{(p_1)} = Z(P_0)$.

Case (B) Since $G_0^{(g)}$ does not contain 3-elements and $G_0^{(g)} \leq N_{G_0}(P_0) = P_0W_0$, by Lemma 7.2 b) we have that $G_0^{(g)}$ is isomorphic to 1, C_2 or C_{q_0-1} . \square

Now we introduce the following notation to determine $U_{N_G(M^{+1})}$, $U_{N_G(M^{-1})}$ and $U_{N_G(M)}$.

Notation 7.9. Let G_0 be a maximal subgroup of G isomorphic to $R(q_0)$. Let $M_0 \in \text{Hall}_{\frac{q_0+1}{4}}(G_0)$. Since $q = q_0^a$, where a is an odd prime, thus $q_0 + 1 \mid q + 1$, and hence $M_0 \leq M$ for some $M \in \text{Hall}_{\frac{q+1}{4}}(G)$. By Theorem 2.1 (e) we have that $C_{G_0}(M_0) = M_0 \times V_0$ and $C_G(M) = M \times V$, for some Klein-subgroups V_0 and V . Since $V \triangleleft N_G(M) = V \times (M \rtimes C_2) \rtimes C_3$, we have that V is contained in every Sylow 2-subgroup of $N_G(M)$. Since by Theorem 2.1 (f), $N_{G_0}(M_0) \leq N_G(M)$, at least one of these Sylow 2-subgroups is inside $N_{G_0}(M_0)$. Hence $V \leq N_{G_0}(M_0)$. Using $[V, M_0] = 1$, we have that $V \leq C_{G_0}(M_0)$ and thus $V_0 = V$. We have that $N_{G_0}(M_0) = (M_0 \times V_0) \rtimes \langle t \rangle$, where the order of element t is 6. Since by Theorem 2.1 (f), $N_G(M_0) \leq N_G(M)$, there is equality here. Hence we have that $N_{G_0}(M_0) \leq N_G(M_0) = N_G(M) = (M \times V_0) \rtimes \langle t \rangle$.

Let $M_0^j \leq G_0$ be a Hall subgroup of order $q_0 + 1 + 3jm_0$, where $j = \pm 1$. It can be embedded to a Hall subgroup $\tilde{M}^{j'}$ of order $q + 1 + j'3m$ in G or to a Hall subgroup \tilde{M} of order $\frac{q+1}{4}$ depending on which factor of $\frac{q^3+1}{4} = \frac{q+1}{4}(q+1+3m)(q+1-3m)$ is divisible by $q_0 + 1 + j3m_0$.)

Furthermore, similarly as in the case of M_0 , by Theorem 2.1 (d) (e) and (f), $N_{G_0}(M_0^j) = M_0^j \rtimes \langle t \rangle$ for $j = \pm 1$ and for an element t of order 6, and $N_G(M_0^j) = N_G(\tilde{M}^{j'})$, where $j' = \pm j$ or $N_G(M_0^j) =$

$N_G(\tilde{M})$, depending on if $M_0^j \leq \tilde{M}^{\pm j}$ or $M_0^j \leq \tilde{M}$. Thus
 $N_G(M_0^j) \in \{\tilde{M}^{+1} \rtimes \langle t \rangle, \tilde{M}^{-1} \rtimes \langle t \rangle, (\tilde{M} \times V) \rtimes \langle t \rangle\}$.

Proposition 7.10. *Using Notation 7.9 we have that if $G_0^{(g)}$ contains a nontrivial element from M_0^{+1} then $g \in G_0 N_G(M_0^{+1})$. In particular*

$U_{N_{G_0}(M_0^{+1})} = \{M_0^{+1}, M_0^{+1} \rtimes C_2\} \cup U$, if $q_0 + 1 + 3m_0 | q + 1$ and

$U_{N_{G_0}(M_0^{+1})} = \{M_0^{+1}\} \cup U$, otherwise,

where U may contain only cyclic subgroups of order 2 or 1.

Proof. Assume that $g \in G$ such that $G_0^{(g)}$ contains a nontrivial element $m \in M_0^{+1}$. Then by Lemma 7.2 c), $M_0^{+1} \leq G_0^{(g)}$, thus $(M_0^{+1})^{g^{-1}} \leq G_0$. Using the fact that the Hall subgroups of order $q_0 + 1 + 3m_0$ are conjugate in G_0 , we have that there is an element $r \in G_0$ such that $(M_0^{+1})^{g^{-1}r} = M_0^{+1}$ and thus $r^{-1}g \in N_G(M_0^{+1})$. Hence $g \in G_0 N_G(M_0^{+1})$.

To prove the second part, suppose that the element $g \in G$ has the property that $G_0^{(g)} \leq N_{G_0}(M_0^{+1})$. We have seen in Lemma 7.8, that if $G_0^{(g)}$ contains a nontrivial 3-element, then it contains the center of a Sylow 3-subgroup of G_0 , thus it also contains a subgroup isomorphic to C_3^2 . This contradicts our assumption $G_0^{(g)} \leq N_{G_0}(M_0^{+1})$. Hence in this case $G_0^{(g)}$ cannot contain nontrivial 3-elements. If $G_0^{(g)}$ does not contain nontrivial elements, whose order divides $q_0 + 1 + 3m_0$, this subgroup can be isomorphic either to C_2 or to $\{1\}$. Assume that $G_0^{(g)}$ contains nontrivial elements from M_0^{+1} . By the first part of the proof, we have that there is an element $r \in G_0$ such that $r^{-1}g \in N_G(M_0^{+1})$. Using Lemma 7.1 b) with $H = N_G(M_0^{+1})$, $t = r$ and $T_0 = G_0$ we have that $G_0^{(g)} = (G_0 \cap N_G(M_0^{+1}))^{(r^{-1}g)} = (N_{G_0}(M_0^{+1}))^{(r^{-1}g)}$, where $r^{-1}g \in N_G(M_0^{+1}) \setminus G_0$. Then $M_0^{+1} \leq N_{G_0}(M_0^{+1})^{(r^{-1}g)}$ by the choice of $r^{-1}g$. Recall that $N_{G_0}(M_0^{+1}) = M_0^{+1} \rtimes \langle t_1 \rangle$, where t_1 is an element of order 6. Observe that both $N_{G_0}(M_0^{+1})$ and $N_{G_0}(M_0^{+1})^{r^{-1}g}$ are Frobenius groups with the same Frobenius kernel. Thus $N_{G_0}(M_0^{+1})^{(r^{-1}g)} \neq M_0^{+1}$, if and only if there is an element $r_1 \in M_0^{+1}$ such that

$$\langle t_1 \rangle^{r^{-1}gr_1} \cap \langle t_1 \rangle \neq 1, \quad (*)$$

where $r^{-1}gr_1 \in N_G(M_0^{+1}) \setminus N_{G_0}(M_0^{+1})$. Depending on the relation of q_0 and q , the subgroup M_0^{+1} is contained in one of \tilde{M}^{+1} , \tilde{M}^{-1} or \tilde{M} .

First assume that $M_0^{+1} \leq \tilde{M}^{+1}$. Then $N_G(M_0^{+1}) = \tilde{M}^{+1} \rtimes \langle t_1 \rangle$ is also a Frobenius group with Frobenius complement $\langle t_1 \rangle$. The equation (*) implies that $r^{-1}gr_1 \in \langle t_1 \rangle$, which is a contradiction. Thus $G^{(g)} = M_0^{+1}$ in this case. This can really occur, e.g. if we choose $g \in \tilde{M}^{+1} \setminus G_0$.

If $M_0^{+1} \leq \tilde{M}^{-1}$, then the proof is similar.

Finally let us assume that $M_0^{+1} \leq \tilde{M}$. Thus $N_G(M_0^{+1}) = (\tilde{M} \times V) \rtimes \langle t_1 \rangle$. Let m be a generator of \tilde{M} and $r^{-1}gr_1 = t_1^a m^b v_1$, where $a, b \in \mathbb{Z}$, and $v_1 \in V \setminus \{1\}$. Suppose that the 3-element t_1^2 acts on $V = \{1, v_1, v_2, v_3\}$ as $v_1^{t_1^2} = v_2$, $v_2^{t_1^2} = v_3$ and $v_3^{t_1^2} = v_1$. The involution t_1^3 centralizes V , thus $v_1^{t_1^3} = v_3$, $v_3^{t_1^3} = v_2$, $v_2^{t_1^3} = v_1$. We have that $t_1^{t_1^a m^b v_1} = t_1^{m^b v_1} = (t_1[t_1, m^b])^{v_1} = t_1^{v_1}[t_1, m^b] = t_1[t_1, v_1][t_1, m^b]$. By Theorem 2.1 (d), $[t_1, m^b] \neq 1$ if and only if $\frac{q+1}{4} \nmid b$. Thus $\langle t_1 \rangle^{t_1^a m^b v_1} \cap \langle t_1 \rangle = \{1\}$, if $\frac{q+1}{4} \nmid b$. However, $\langle t_1 \rangle^{t_1^a v_1} = \langle t_1 \rangle^{v_1}$ and since $t_1^{v_1} = v_3 t_1$, $(t_1^{v_1})^2 = v_2 t_1^2$, $(t_1^{v_1})^3 = t_1^3$, we have that $\langle t_1 \rangle^{t_1^a v_1} \cap \langle t_1 \rangle = \langle t_1^3 \rangle$.

Thus if $G_0^{(g)}$ contains nontrivial elements from M_0^{+1} and $M_0^{+1} \leq \tilde{M}$ then $G^{(g)}$ can be either M_0^{+1} or $M_0^{+1} \rtimes \langle t_1^3 \rangle$. These cases in fact occur. If $g \in \tilde{M} \setminus M_0$, then $G_0^{(g)} = M_0^{+1}$ and if $g \in V \setminus \{1\}$, then $G_0^{(g)} = M_0^{+1} \rtimes \langle t_1^3 \rangle$. \square

Proposition 7.11. *Using Notation 7.9 we have that if $G_0^{(g)}$ contains a nontrivial element from M_0^{-1} , then $g \in G_0 N_G(M_0^{-1})$. In particular*

$U_{N_{G_0}(M_0^{-1})} = \{M_0^{-1}, M_0^{-1} \rtimes C_2\} \cup U$, if $q_0 + 1 - 3m_0 | q + 1$ and

$U_{N_{G_0}(M_0^{-1})} = \{M_0^{-1}\} \cup U$, otherwise,

where U may contain only cyclic subgroups of order 2 or 1.

Proof. The proof is similar to the previous one. \square

Proposition 7.12. *Using Notation 7.9 we have that if $G_0^{(g)}$ contains a nontrivial element from M_0 , then $g \in G_0 N_G(M_0)$. In particular*

$U_{N_{G_0}(M_0)} = \{M_0 \times V\} \cup U$,

where U may contain only cyclic subgroups of order 2 or 1.

Proof. Assume that $g \in G$ such that $G_0^{(g)}$ contains a nontrivial element $m \in M_0$. Then by Lemma 7.2 a) we have that $M_0 \leq G_0^{(g)}$, hence $M_0, M_0^{g^{-1}} \leq G_0$. Since the Hall subgroups of order $\frac{q_0+1}{4}$ are conjugate in G_0 , there exists an element $r \in G_0$ such that $M_0^{g^{-1}r} = M_0$ and thus $r^{-1}g \in N_G(M_0)$. Hence $g \in G_0 N_G(M_0)$.

To prove the second part, suppose that the element $g \in G$ has the property that $G_0^{(g)} \leq N_{G_0}(M_0)$ holds. We have seen in Lemma 7.7, that if a nontrivial 3-element lies in $G_0^{(g)}$, then $G_0^{(g)}$ contains the center of a Sylow 3-subgroup of G_0 , and hence it also contains a subgroup isomorphic to C_3^2 . This contradicts our assumption $G_0^{(g)} \leq N_{G_0}(M_0)$. Thus $G_0^{(g)}$ cannot contain nontrivial 3-elements. If $G_0^{(g)}$ does not contain elements, whose order divides $\frac{q_0+1}{4}$, then it can only be isomorphic to C_2^3 , C_2^2 , C_2 or $\{1\}$. By Lemma 7.2 d) we know that C_2^2 and C_2^3 cannot happen. Assume that $G_0^{(g)}$ contains a nontrivial element from M_0 . By the first part of the proof, we have that there is an element $r \in G_0$ such that $r^{-1}g \in N_G(M_0)$. Using Lemma 7.1 b) with $T_0 = G_0$, $H = N_G(M_0)$ and $t = r$, we have that

$G_0^{(g)} = (G_0 \cap N_G(M_0))^{(r^{-1}g)} = (N_{G_0}(M_0))^{(r^{-1}g)}$, where $r^{-1}g \in N_G(M_0) \setminus G_0$. By Lemma 7.2 a) we have that $M_0 \times V \leq (N_{G_0}(M_0))^{(r^{-1}g)}$. We want to prove that there is equality here. Recall that $N_{G_0}(M_0) = (M_0 \times V) \rtimes \langle t \rangle$, where t is an element of order 6 and $N_G(M_0) = (M \times V) \rtimes \langle t \rangle$. If $N_{G_0}(M_0) \cap N_{G_0}(M_0)^{r^{-1}g} > M_0 \times V$ holds, then this intersection contains $\langle t^k \rangle^x$ for some integer $0 < k < 6$ and $x \in M_0 \times V$. Hence it also contains $\langle t^k \rangle$. But then $\langle t^k \rangle \leq \langle t \rangle^{r^{-1}gy}$, for some $y \in M_0 \times V$. Since $r^{-1}g \in N_G(M_0)$, $r^{-1}gy = t^s m v$ for some integer s , $v \in V$ and $m \in M \setminus M_0$. Thus $\langle t^k \rangle \leq \langle t \rangle^{mv}$ and hence $\langle t^k \rangle = \langle t^k \rangle^{mv}$. Then $[mv, t^k] \in (M \times V) \cap \langle t^k \rangle = 1$. Thus $t^{km} = t^{kv}$ and hence $t^{km^2} = t^k$. Since by Lemma 7.2 a), $C_G(m^2) = M \times V$ and $0 < k < 6$, this cannot happen and we have $G_0^{(g)} = N_{G_0}(M_0)^{(r^{-1}g)} = M_0 \times V$.

This case in fact occurs. Let $g \in M \setminus M_0$. Then $G_0^{(g)} \neq G_0$ and $M_0 \times V \leq G_0^{(g)}$. Thus, $G_0^{(g)} \leq N_{G_0}(M_0)$, or $G_0^{(g)} \leq C_{G_0}(i)$, for some involution $i \in G_0$. By the results above, in the first case $G_0^{(g)} = M_0 \times V$. We will show that the second case alone does not occur. Let us suppose by contradiction that $M_0 \times V \leq G_0^{(g)} \leq C_{G_0}(i)$. Since $g \in M \leq C_G(i)$, thus $i^g = i$. By Lemma 7.1 b) with $T_0 = G_0$, $H = C_G(i)$ and $t = 1$, we have that $G_0^{(g)} = (G_0 \cap C_G(i))^{(g)} = C_{G_0}(i)^{(g)}$. Let $C_{G_0}(i) = \langle i \rangle \times L$, where $L \simeq PSL(2, q_0)$.

By a result of Levchuk and Nuzhin, see [21, Lemmas 5 and 6], if $G_0^{(g)}$ is solvable then in our case it is contained in $N_{G_0}(M_0)$, and we are done. If it is nonsolvable, then since it contains $M_0 \times V$ it can only be $C_{G_0}(i) = \langle i \rangle \times L$ from the possible list of subgroups. Thus $C_{G_0}(i)^{(g)} = C_{G_0}(i)$. However, by Proposition 7.8, we know that $G_0^{(g)} = C_G(i)^{(g)}$ must contain the center of a Sylow 3-subgroup, which by Theorem 2.1 (b) cannot happen. \square

Proposition 7.13. *Let R_0 be a maximal subgroup in G_0 isomorphic to $R(q_1)$. Then U_{R_0} may contain some cyclic subgroups of order 2 or 1.*

Proof. Let $g \in G$ be such that $G_0^{(g)} \leq R_0$. If $G_0^{(g)}$ would contain nontrivial elements whose orders divide either $\frac{q_0+1}{4}$, $\frac{q_0-1}{2}$ or $q_0 \pm 3m_0 + 1$, then by Lemma 7.2 a), b) and c) it would contain a subgroup isomorphic to $C_2^2 \times C_{\frac{q_0+1}{4}}$, C_{q_0-1} or $C_{q_0 \pm 3m_0 + 1}$, respectively. Hence $G_0^{(g)}$ could not be inside R_0 . Thus in $G_0^{(g)}$ a nontrivial element must have order divisible by 2 or 3. If $G_0^{(g)}$ contains a nontrivial 3-element, then by Lemma 7.8 a) $Z(P_0) \simeq C_3^{2n_0+1} \leq G_0^{(g)}$ and so $G_0^{(g)}$ cannot be inside R_0 . Furthermore, $G_0^{(g)}$ cannot contain more than one involution. Otherwise, if it contains two commuting involutions, then by Lemma 7.2 d) $C_2^2 \times C_{\frac{q_0+1}{4}} \leq G_0^{(g)}$, which is a contradiction. If $G_0^{(g)}$ contains two non-commuting involutions, then they generate a dihedral group. It either contains a Klein four subgroup, or an element of odd order. None of them can happen. Thus $G_0^{(g)}$ can only be $\{1\}$ or a cyclic subgroup of order 2. \square

Now we determine $U_{C_{G_0}(i)}$.

Proposition 7.14. *Let i be an involution in G_0 . Then*

$$U_{C_{G_0}(i)} = \{1, \langle k \rangle, W_0, M_0 \times V \mid \langle k \rangle \simeq C_2, [i, k] = 1, i \in W_0 \simeq C_{q_0-1}, i \in V, M_0 \times V \simeq C_{\frac{q_0+1}{4}} \times C_2^2\}.$$

Morover, every cyclic subgroup of order 2 and $q_0 - 1$ in G_0 occurs as $G_0^{(x)}$ for some suitable $x \in G$.

Proof. Let $g \in G$ be such that $G_0^{(g)} \leq C_{G_0}(i)$. Let us suppose that $p \in G_0^{(g)}$ is an element of order 3 and denote by P_0 the Sylow 3-subgroup of G_0 , which contains it. Then by Lemma 7.8 we have that $Z(P_0) \leq G_0^{(g)} \leq C_{G_0}(i)$. However, by Theorem 2.1 (b), $Z(P_0) \cap C_{G_0}(i) = \{1\}$, which is a contradiction.

Hence, since $C_{G_0}(i) \simeq \langle i \rangle \times PSL(2, q_0)$, the only prime order elements in $G_0^{(g)}$ are of order 2 and of orders that are divisors of $\frac{q_0-1}{2}$ or $\frac{q_0+1}{4}$.

By Lemma 7.2 a), b) and d) we know that $G_0^{(g)}$ is a cyclic subgroup of order 2 or contains either a cyclic subgroup of order $q_0 - 1$ or a subgroup isomorphic to $C_{\frac{q_0+1}{4}} \times C_2^2$.

By Theorem 6.1 about the list of subgroups of $PSL(2, q_0)$, the subgroups of $C_{G_0}(i)$, which can contain C_{q_0-1} or $C_{\frac{q_0+1}{4}} \times C_2^2$ and do not contain nontrivial 3-elements are isomorphic to C_{q_0-1} , $D_{q_0-1} \times C_2$, $C_{\frac{q_0+1}{4}} \times C_2^2$ or $(C_{\frac{q_0+1}{4}} \times C_2^2) \rtimes C_2$.

On the other hand, if $G_0^{(g)} \leq N_{G_0}(M_0)$, then by Proposition 7.12, the subgroup $G_0^{(g)}$ can be only 1, C_2 or $M_0 \times V$. Since $(C_{\frac{q_0+1}{4}} \times C_2^2) \rtimes C_2$ is isomorphic to a subgroup of $N_{G_0}(M_0)$, we may exclude it from being isomorphic to $G_0^{(g)}$.

Moreover, according to Lemma 7.2 d) we can also exclude that $G_0^{(g)} \simeq D_{q_0-1} \times C_2$. Summarizing: $G_0^{(g)}$ is isomorphic to one of the following groups: 1, C_2 , C_{q_0-1} , $C_{\frac{q_0+1}{4}} \times C_2^2$.

We know, by Proposition 7.12, that the subgroups of $C_{G_0}(i)$ isomorphic to $C_{\frac{q_0+1}{4}} \times C_2^2 \simeq M_0 \times V$ occur as $G_0^{(g)}$ (we may suppose that $g \in M \setminus M_0$ and $i \in V$). Now we give a construction for the remaining three cases.

Denote by L the factor group $C_G(i)/\langle i \rangle$. Let L_0 be the image of $C_{G_0}(i)$ under the natural homomorphism $\pi : C_G(i) \rightarrow L$. By [4, Lemma 2.16], there exists an element $\bar{l}_1 \in L$ such that $L_0^{(\bar{l}_1)} = \{1\}$, since $2(2n_0 + 1) < 2n + 1$. We will need that more than $\frac{1}{2}q_0^2(q_0 - 1)^2(q_0 + 1)^2$ such elements exist. It is shown in the proof of [4, Lemma 2.16] the number of them is at least $A = \frac{q(q-1)(q+1) - q q_0^4 - 2q q_0^3 - 3q q_0^2 + 2q q_0 + 2q + 2q_0^5 + 3q_0^4 - 2q_0^3 - q_0^2}{2}$. If we use that $q_0 \leq q/9$, $q_0^2 \leq q/3$, $q_0^3 \leq q$ and $(2q q_0 + 2)q_0^5 + 5q_0^4 - 2q_0^3 - 2q_0^2 \geq 0$, then we have that $A - \frac{1}{2}q_0^2(q_0 - 1)^2(q_0 + 1)^2 \geq \frac{q^3 - q - q^3/9 - 2q^2 - 3q^2/3 + 2q - q^2}{2} = \frac{8/9 q^3 - 4q^2 + q}{2}$. Since it is bigger than 0, if $q \geq 27$, we get that there are more than $\frac{1}{2}q_0^2(q_0 - 1)^2(q_0 + 1)^2$ elements $\bar{l}_1 \in L$ such that $L_0^{(\bar{l}_1)} = \{1\}$. Hence, by taking inverse images, we have that there are more than $q_0^2(q_0 - 1)^2(q_0 + 1)^2$ elements $l_1 \in C_G(i)$ such that $C_{G_0}(i)^{(l_1)} = \langle i \rangle$.

We want to prove that there exists an element $\bar{l}_2 \in L$ such that $L_0^{(\bar{l}_2)} \simeq C_{\frac{q_0-1}{2}}$. Let $U_0 \leq L_0$ be a subgroup isomorphic to $C_{\frac{q_0-1}{2}}$. By [4, Theorem 1.2 (ii)] we have that $N_L(U_0) \simeq D_{q-1}$ and $N_{L_0}(U_0) \simeq D_{q_0-1}$. Let us denote by U the maximal cyclic subgroup of $N_L(U_0)$. Since L_0 is selfnormalizing in L , by the proof of [4, Lemma 2.16], thus if we take an element \bar{l}_2 from $U \setminus U_0$ then $U_0 \leq L_0^{(\bar{l}_2)} \neq L_0$. Hence, by [4, Lemma 2.15], we have that $U_0 = L_0^{(\bar{l}_2)}$. Taking an inverse image l_2 of \bar{l}_2 , we have that $(C_{G_0}(i))^{(l_2)} \simeq \langle i \rangle \times C_{\frac{q_0-1}{2}}$. We show that if $G_0^{(l_1)}$ and $G_0^{(l_2)}$ are subgroups of $C_{G_0}(i)$, then they are isomorphic to C_2 and C_{q_0-1} , respectively. Applying Lemma 7.1 c) with $H = \langle i \rangle$, $r = 1$ and $g = l_1$ or $g = l_2$, respectively, we have that $G_0^{(l_1)} \cap C_{G_0}(i) = G_0^{(l_1)} \cap N_G(\langle i \rangle) = N_{G_0}(\langle i \rangle)^{(l_1)} = C_{G_0}(i)^{(l_1)} = \langle i \rangle$ and $G_0^{(l_2)} \cap C_{G_0}(i) = C_{G_0}(i)^{(l_2)} \simeq C_{q_0-1}$. If $G_0^{(l_1)}$ and $G_0^{(l_2)}$ are contained in $C_{G_0}(i)$, then $G_0^{(l_1)} = \langle i \rangle$ and $G_0^{(l_2)} \simeq C_{q_0-1}$, containing i . For another involution $k \in C_{G_0}(i)$ we may choose an element $x \in G_0$ such that $i^x = k$, then $\langle k \rangle = G_0^{(l_1 x)}$. So we are done with the first statement of the Proposition in this case. Let us suppose that we do not know if $G_0^{(l_1)}$ and $G_0^{(l_2)}$ are inside $C_{G_0}(i)$. Since both $G_0^{(l_1)}$ and $G_0^{(l_2)}$ contain i , by Proposition 7.7, 7.10, 7.11, 7.12 and 7.13, depending on which maximal subgroup of G_0 contains them, they are isomorphic to one of the following: C_2 , C_{q_0-1} , $P_0' \rtimes C_2$, $P_0 \rtimes C_2$, $P_0 \rtimes C_{q_0-1}$, $C_2^2 \times C_{\frac{q_0+1}{4}}$, $C_{q_0+3m_0+1} \rtimes C_2$, $C_{q_0-3m_0+1} \rtimes C_2$ or G_0 . The intersection of one of these subgroups with the centralizer $C_{G_0}(i)$ of one of their involutions i , is isomorphic to the following: C_2 , C_{q_0-1} , $C_3^{2n_0+1} \rtimes C_2$, $C_3^{2n_0+1} \rtimes C_2$, $C_3^{2n_0+1} \rtimes C_{q_0-1}$, $C_2^2 \times C_{\frac{q_0+1}{4}}$, C_2 , C_2 , $C_{G_0}(i)$. Since this intersection must be C_2 or C_{q_0-1} , this implies that the only possibilities are: $G_0^{(l_2)} \simeq C_{q_0-1}$, $G_0^{(l_1)} = \langle i \rangle$ or $G_0^{(l_1)} = M_0^{\pm 1} \rtimes \langle i \rangle$ for some $M_0^{\pm 1} \in \text{Hall}_{q_0 \pm 3m_0+1}(G_0)$. In Proposition 7.10 and 7.11 we have seen that if $G_0^{(g)} \simeq C_{q_0 \pm 3m_0+1} \rtimes C_2$, then $(M_0)^{\pm 1} \leq \tilde{M} \in \text{Hall}_{\frac{q_0+1}{4}}(G)$ and $g \in G_0(V \setminus \{1\})$, where $C_2^2 \simeq V \leq C_G(\tilde{M})$. There are $\frac{|C_{G_0}(i)|}{|N_{C_{G_0}(i)}(M_0^{\pm 1})|} = \frac{q_0(q_0-1)(q_0+1)}{6}$ subgroups in G_0 , being isomorphic to $C_{q_0 \pm 3m_0+1}$ and having the property, that the normalizer contains i . Since $V \leq C_G(i)$, there are at most $2 \frac{q_0(q_0-1)(q_0+1)}{6} |C_{G_0}(i)V \setminus 1| = q_0^2(q_0 - 1)^2(q_0 + 1)^2$ elements $g \in C_G(i)$ such that $G_0^{(g)} \simeq C_{q_0 \pm 3m_0+1} \rtimes C_2$. However, we have seen that there are more elements $l_1 \in C_G(i)$ such

that $C_{G_0}(i)^{(l_1)} = \langle i \rangle$. This implies that there are elements $l_1, l_2 \in C_G(i)$ such that $G_0^{(l_1)} = \langle i \rangle$ and $G_0^{(l_2)} \simeq C_{q_0-1}$.

Since both the involutions and cyclic subgroups of order $q_0 - 1$ are conjugate in G_0 , every cyclic subgroup of order 2 and $q_0 - 1$ occurs as $G_0^{(x)}$ for some $x \in G$.

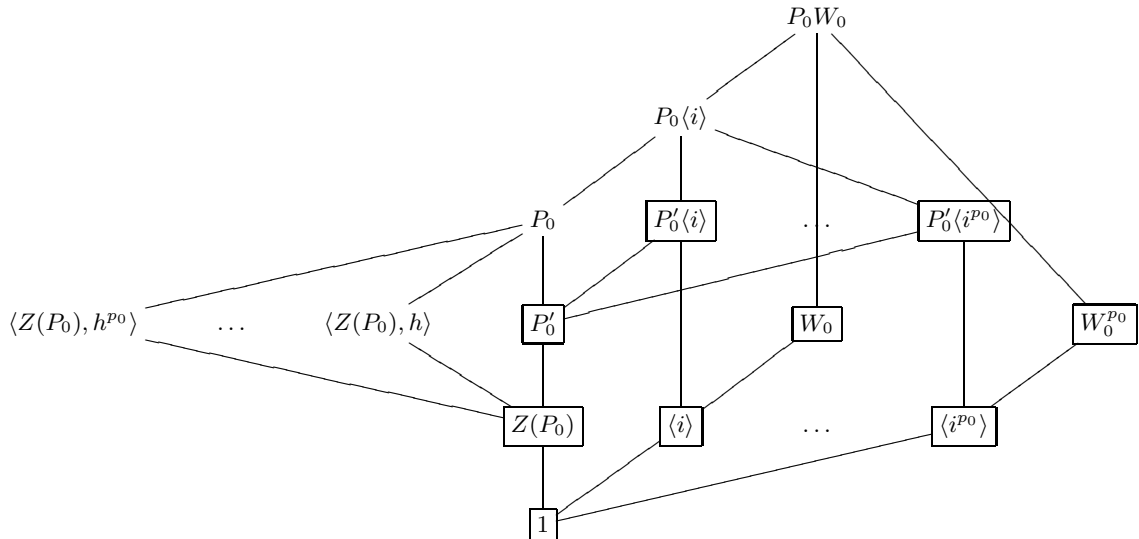
Finally, we will show that there exists an element $g \in G$ such that $G_0^{(g)} = 1$. Looking at the order of G_0 , by Lemma 7.2 it is enough to show, that there is an element g in G such that $G_0^{(g)}$ contains neither an involution $i \in G_0$, nor elements from P_0 and $M_0^{\pm 1}$ for every $P_0 \in \text{Syl}_3(G_0)$ and $M_0^{\pm 1} \in \text{Hall}_{q_0 \pm 3m_0 + 1}(G_0)$. We know that if $G_0^{(g)}$ contains elements from P_0 or $M_0^{\pm 1}$, then by Lemma 7.8, Proposition 7.10 and Proposition 7.11, $g \in G_0 N_G(P)$, (where P is the unique Sylow p -subgroup of G containing P_0) or $g \in G_0 N_G(M_0^{\pm 1})$, respectively. If the involution i is in $G_0^{(g)}$, then using Lemma 7.1 a) with $H_0 = \langle i \rangle$ and $T_0 = G_0$ we have that, $g \in G_0 C_G(i)$. Thus, we can give an upper bound to the cardinality

$$\begin{aligned} & |\cup_{P_0 \in \text{Syl}_3(G_0)} G_0 N_G(P) \cup \cup_{M_0^{\pm 1} \in \text{Hall}_{q_0 \pm 3m_0 + 1}(G_0)} G_0 N_G(M_0^{\pm 1}) \cup \cup_{i \in G_0, o(i)=2} G_0 C_G(i)| \leq \\ & \sum_{P_0 \in \text{Syl}_3(G_0)} |G_0| |N_G(P)| / |N_{G_0}(P)| + \sum_{M_0^{\pm 1} \in \text{Hall}_{q_0 \pm 3m_0 + 1}(G_0)} |G_0| |N_G(M_0^{\pm 1})| / |N_{G_0}(M_0^{\pm 1})| + \\ & \sum_{i \in G_0, o(i)=2} |G_0| |C_G(i)| / |C_{G_0}(i)| = \\ & |G_0| (|G_0 : N_{G_0}(P_0)| \frac{|N_G(P_0)|}{|N_{G_0}(P_0)|} + \sum_{i=\pm 1} |\text{Hall}_{q_0 \pm 3m_0 + 1}(G_0)| \frac{|N_G(M_0^i)|}{|N_{G_0}(M_0^i)|} + [G_0 : C_{G_0}(i)] \frac{|C_G(i)|}{|C_{G_0}(i)|}) \end{aligned}$$

Furthermore, we use that $|\text{Hall}_{q_0 \pm 3m_0 + 1}(G_0)| \leq \frac{q_0^3(q_0^2-1)(q_0+3m_0+1)}{6}$, $|N_G(M_0^{\pm 1})| \leq |N_G(M_0^{+1})| = 6(q+3m+1)$ and $|G_0 \cap N_G(M_0^{\pm 1})| \geq |N_{G_0}(M_0^{\pm 1})| = 6(q_0-3m_0+1)$. Thus the cardinality of the above set is at most $q_0^3(q_0-1)(q_0^3+1) \left((q_0^3+1) \cdot \frac{q^3(q-1)}{q_0^3(q_0-1)} + 2 \frac{q_0^3(q_0^2-1)(q_0+3m_0+1)}{6} \cdot \frac{6(q+3m+1)}{6(q_0-3m_0+1)} + q_0^2(q_0^2 - q_0 + 1) \frac{q(q-1)(q+1)}{q_0(q_0-1)(q_0+1)} \right) = (q_0^3+1)^2 q^3 (q-1) + \frac{1}{3} q_0^6 (q_0^2-1)^2 (q_0+3m_0+1)^2 (q+3m+1) + q_0^4 (q_0^2 - q_0 + 1)^2 q (q-1)(q+1)$. A naive upper bound for this is: $(2q_0^3)^2 q^3 (q-1) + \frac{1}{3} q_0^6 (q_0^2)^2 (3q_0)^2 (3q) + q_0^4 (q_0^2)^2 q (q-1)(2q) = (q-1)(4q_0^6 q^3 + \frac{1}{3} q_0^{12} \frac{q}{q-1} 27 + 2q_0^8 q^2)$. Using that $q > 3$ (actually we have $q \geq 27$), we have that $\frac{q}{q-1} \leq \frac{3}{2}$ and $q_0^3 \leq q$, and so the cardinality of the above set is at most $(q-1)(4q^5 + 27/2 q^4 + 2q^5) \leq 20(q-1)q^5$. For $q \geq 27$ this is obviously smaller than $|G| = (q-1)q^3(q^3+1)$, and as we have seen before, this implies that there is an element $g \in G$ such that $G_0^{(g)} = \{1\}$. \square

Theorem 7.15. *We have that for maximal subgroups $G_0 \simeq R(q_0)$ in $R(q)$, where $q_0 > 3$ $\mathcal{U}_1(G_0) = \mathcal{U}_2(G_0)$, $d_c(G_0, G) = 4$ and $d(G_0, G) = 3$.*

Proof. We use Notations 7.5 and 7.9. By Propositions 7.7, 7.10, 7.11 7.13 and 7.14 we know that $\mathcal{U}_1(G_0) = \bigcup_{P_0 \in \text{Syl}_3(G_0)} U_{N_{G_0}(P_0)} \cup \{H | H \simeq 1, C_2, C_{q_0-1}, C_{q_0+1 \pm 3m_0}, C_2^2 \times C_{\frac{q_0+1}{4}}\}$ and sometimes $C_{q_0+1 \pm 3m_0} \rtimes C_2$ also belong to it. Since $G_0^{(g_1, g_2)} = G_0^{(g_1)} \cap G_0^{(g_2)}$, the set $\mathcal{U}_1(G_0)$ is closed under intersection if and only if $\mathcal{U}_1(G_0) = \mathcal{U}_2(G_0)$. It is easy to see, that $\mathcal{U}_1(G_0)$ will be intersection closed, if $U_{N_{G_0}(P_0)}$ is intersection closed for every $P_0 \in \text{Syl}_3(G_0)$. To show this, we display the subgroups according to Proposition 7.7 and 7.14 framing those subgroups, which are surely part of $\mathcal{U}_1(G_0)$.



It can be checked easily that any subgroups without frame can be skipped without any problem and the remaining set of subgroups is still closed under intersection. Using condition (1) in Definition 1.1, we have that $d_c(G_0, G) \leq 4$. Using condition (2) in Definition 1.1, we show that $d_c(G_0, G) > 3$.

Let $x_2 \in G$ be such that $G_0^{(x_2)} = P'_0$ (actually we can choose any proper subgroup of G_0 from $\mathcal{U}_1(G_0)$ instead of P'_0) and let $x_1 \in G_0 \setminus N_{G_0}(P'_0)$. Then $G_0^{(x_1, x_2)} = P'_0$. Assume by contradiction that there exists an element $y_1 \in G$ such that $G_0^{(x_1, x_2)} = G_0^{(y_1)}$ and $h^{x_1} = h^{y_1}$ for all $h \in P'_0$. Since $P_0^{x_1} = P_0^{y_1}$, we have that $P_0^{x_1} \leq G_0^{y_1}$ and so $P_0^{x_1} \leq G_0^{(y_1)} = P'_0$. By the choice of x_1 , i. e. $x_1 \notin N_G(P'_0)$, we have a contradiction. Thus $d_c(G_0, G) > 3$ and hence $d_c(G_0, G) = 4$. Using Theorem 1.3, we have that $d(G_0, G) = 3$. \square

Remark 7.16. *With similar methods we also proved that if $G_0 \simeq R(3)$ then $d_c(G_0, G) = 4$, $d(G_0, G) = 3$ also holds. Its proof is in the next section.*

8. THE DEPTH OF $H \simeq R(3)$

Let H be a subgroup of $G \simeq R(3^{2n+1})$, $n > 1$ that is isomorphic to $R(3)$. This subgroup is maximal in $G \simeq R(3^{2n+1})$, if $2n + 1$ is prime. Let us list some properties of H . They can be easily checked e.g. in the GAP system, [7].

Proposition 8.1. *a) $|H| = 3^3 \cdot 2^3 \cdot 7 = 1512 = (3^3 + 1) \cdot 3^3 \cdot 2$.*

b) The Sylow 3-subgroups in H form TI sets. Let $P_0 \in \text{Syl}_3(H)$. Then $Z(P_0) = P'_0 \simeq C_3$. The normalizer of P_0 is $N_H(P_0) = P_0 \rtimes \langle i \rangle$, where i is an involution. We also have that $C_{P_0}(i) \simeq C_3$ and $C_{P_0}(i) \cap Z(P_0) = \{1\}$. Denote by P_0^1 the elementary abelian subgroup generated by the elements of order 3 in P_0 . All elements of $P_0 \setminus P_0^1$ are of order 9 and their 3-rd powers belong to $Z(P_0)$. P_0 has a representation in triples, similar to P , $\{(x, y, z) | x, y, z \in GF(3)\}$ and σ acts identically on $GF(3)$. $P'_0 = Z(P_0) = \{(0, 0, z) | z \in GF(3)\}$, $P^1 = P' \cap P_0 = \{(0, y, z) | y, z \in GF(3)\}$.

c) Let $S \in \text{Syl}_2(H)$. Then S is elementary abelian group of order 8, $|N_H(S)| = 8 \cdot 7 \cdot 3$ and $C_G(S) = S$. The 2-subgroups of equal order are conjugate in H . The centralizer of an involution $i \in H$ is $C_H(i) \simeq \langle i \rangle \times A_4$. It is contained in $N_H(S)$ for some $S \in \text{Syl}_2(H)$.

d) The Sylow 7-subgroups in H form TI sets. Let $M_0^{+1} \in \text{Syl}_7(G)$. Then the subgroup $N_H(M_0^{+1})$ is a Frobenius group with kernel M_0^{+1} and cyclic complement of order 6.

e) Each maximal subgroup of H is conjugate to one of the following:

$N_H(P_0)$, $N_H(S)$, $N_H(M_0^{+1})$ or it equal to $H' \simeq PSL(2, 8)$.

Their orders are: 54, 168, 42 and 504, respectively.

We will prove the following Lemma for $R(3)$, which is similar to Lemma 7.8 and Proposition 7.12 for $R(q_0)$.

Lemma 8.2. *Let $g \in G$ be an element such that $H^{(g)} \neq H$. Let $P_0 \in \text{Syl}_3(H)$ and $P_0 \leq P \in \text{Syl}_3(G)$. Further let $N_H(P_0) = P_0 \rtimes I$, where $I \simeq C_2$ and let $N_G(P) = P \rtimes W$ such that $I \leq W$. Then*

(A) If $H^{(g)}$ contains a 3-element $p \in P_0$, then there exists an element $r \in H$ such that $r^{-1}g \in N_G(P)$. Moreover, this element r can be chosen so that $r^{-1}g = p_1 \in P \setminus P_0 = (Z(P) \setminus Z(P_0)) \cup (P' \setminus P_0^1 Z(P)) \cup (P \setminus P_0 P')$ and $H^{(g)} = (P_0 I)^{(r^{-1}g)} = (P_0 I)^{(p_1)}$. In particular, $Z(P_0) \leq H^{(g)}$ and $H^{(g)} \leq N_H(P_0)$.

I. If $r^{-1}g \in Z(P) \setminus Z(P_0)$ then $H^{(g)} = P_0$ or $P_0 \langle i \rangle$.

II. If $r^{-1}g \in P' \setminus P_0^1 Z(P)$ then $H^{(g)}$ is isomorphic to P_0^1 or $P_0^1 \langle i \rangle$. Moreover, there exists an element $p_1 \in P' \setminus P_0^1 Z(P)$ such that $H^{(p_1)} = P_0^1$ and for every involution $j \in P_0 I$ there exists an element $p_2 \in P' \setminus P_0^1 Z(P)$ such that $H^{(p_2)} = P_0^1 \langle j \rangle$.

III. If $r^{-1}g \in P \setminus P_0 P'$ then $H^{(g)}$ is $Z(P_0)$ or $Z(P_0) \langle h \rangle$, where h is an element of order 9 in P_0 . Moreover, there exists an element $p_1 \in P \setminus P_0 P'$ such that $H^{(p_1)} = Z(P_0)$.

(B) If $H^{(g)} \leq N_H(P_0)$ and $H^{(g)}$ does not contain any nontrivial 3-elements from P_0 , then $H^{(g)}$ is isomorphic to 1 or C_2 .

Proof. (A) Let $p \in H^{(g)}$ be a nontrivial 3-element belonging to P_0 . Let $S_0 \in \text{Syl}_3(H)$ containing $p^{g^{-1}}$. Then there exists an element $r \in H$ such that $S_0^r = P_0$. Since $p, p^{g^{-1}r} \in P_0 \leq P$, and P is TI, we have that $r^{-1}g \in N_G(P)$. Thus $g \in HN_G(P) \setminus H$. Then, since $HW P = HIP \cup H(W \setminus I)P = H(P \setminus P_0) \cup H \cup H(W \setminus I)P$,

we have that

$$HN_G(P) \setminus H = H(P \setminus P_0) \cup H(W \setminus I)P.$$

We will show that if $g \in H(W \setminus I)P$ then $H^{(g)}$ does not contain any non-trivial elements from P_0 . Thus in part (A) $g \in H(P \setminus P_0)$ must hold.

Let $g = rw_1 p_1$, where $r \in H$, $w_1 \in W \setminus I$ and $p_1 \in P$. We may suppose that $w_1 \in W_2$, otherwise, since $g = (ri)(iw_1)p_1$, we choose $ri \in H$ instead of r and $i w_1 \in W \setminus I$ instead of w_1 . Using the fact that $N_H(P_0) \leq N_H(P)$ and Lemma 7.1 c), with $G_0 := H$, $H := P$ and $r := r$, we have that $H^{(g)} \cap N_G(P) = N_H(P)^{(w_1 p_1)} = N_H(P_0)^{(w_1 p_1)} = (P_0 I)^{(w_1 p_1)}$.

Let $1 \neq z \in Z(P_0) \cap (P_0 I)^{w_1 p_1}$. Then $z \in (P_0 I)^{w_1}$ and $z^{w_1^{-1}} \in P_0 \cap Z(P) = Z(P_0)$. Let $z_0 = z^{w_1^{-1}}$. Then there exists an element $w_0 \in I$ such that $z^{w_0} = z_0$. Hence $z^{w_0 w_1} = z$. Using Lemma 7.3 a) we have a contradiction. Hence $(P_0 I)^{(w_1 p_1)}$ does not contain any nontrivial elements from $Z(P_0)$. Thus $P_0 I^{(w_1 p_1)}$ neither contains any elements of order 9 from P_0 . We show by contradiction that $(P_0 I)^{(w_1 p_1)}$ does not contain any other 3-elements. Let h be an element of order 3 in $(P_0 I)^{(w_1 p_1)}$. Then $h \in P_0 \setminus Z(P_0)$. Using the representation of the Sylow 3-subgroups $P_0 \in \text{Syl}_3(H)$ and $P \in \text{Syl}_3(G)$, we have that $h = (0, y_0, z_0)$ and $p_1^{-1} = (a, b, c)$, where $y_0 \neq 0$, $y_0, z_0 \in GF(3)$ and $a, b, c \in GF(q)$. Denote w_1^{-1} by w . Then by Theorem 2.1 h) and (by Lemma 2.6) we have that

$$h^{p_1^{-1} w} = (0, y_0, z_0)^w (0, 0, 2y_0 a)^w \in P_0 \cap P' = P_0^1.$$

Denote the element $(0, 0, 2y_0 a)^w$ by ζ . Since $(0, y_0, z_0)^w \zeta \in P_0^1 \setminus Z(P_0)$, there exists an element $\zeta_0 \in Z(P_0)$ such that $(0, y_0, z_0)^w \zeta = (0, y_0, z_0) \zeta_0$ or $(0, y_0, z_0)^w \zeta = (0, y_0, z_0)^{-1} \zeta_0$. Using Lemma 7.3 b) for P' , we have in both cases a contradiction. In the first case $w \in W_{2'}$ fixes $(0, y_0, z_0)Z(P)$, and in the second case w takes $(0, y_0, z_0)Z(P)$ to its inverse. Thus $H^{(g)} \cap P_0 = \{1\}$, which is a contradiction. Thus $g \notin H(W \setminus I)P$.

Hence if $H^{(g)}$ contains a nontrivial 3-element $p \in P_0$, then $g \in H(P \setminus P_0)$ i. e. $g = r p_1$ where $r \in H$ and $p_1 \in P \setminus P_0$. Thus by Lemma 7.1 d) with H as G_0 and $Z(P_0)$ as H_0 , we have that $H^{(g)} = (N_H(Z(P_0)))^{(p_1)} = (P_0 I)^{(p_1)}$.

Obviously, $P \setminus P_0$ is a disjoint union of $Z(P) \setminus Z(P_0)$, $P' \setminus P_0^1 Z(P)$ and $P \setminus P_0 P'$. Hence $p_1 = r^{-1} g$ belongs to one of them.

- I. If $r^{-1} g = p_1 \in Z(P) \setminus Z(P_0)$ then as $[P_0, p_1] = 1$, $H^{(g)} = (P_0 I)^{(p_1)} \geq P_0$. Thus we have that $H^{(g)}$ is P_0 or $P_0 I$.
- II. If $r^{-1} g = p_1 \in P' \setminus P_0^1 Z(P)$, then $H^{(g)} = (P_0 I)^{(p_1)} \geq P_0^1$, since P' is elementary abelian. If $H^{(g)} = (P_0 I)^{(p_1)}$ would contain an element h of order 9, then $h^{p_1^{-1}} \in P_0$ would hold. Using the representation of elements of the Sylow 3-subgroup of G in Theorem 2.1 h) and Lemma 2.6, we have that if $p_1^{-1} = (0, b, c)$ and $h = (x_0, y_0, z_0)$ with $b \notin GF(3)$, $x_0 \neq 0$ and $x_0, y_0, z_0 \in GF(3)$, then $h^{p_1^{-1}} = (x_0, y_0, z_0 - 2x_0 b) \in P_0$, which is a contradiction. Thus $H^{(g)}$ can only be isomorphic to P_0^1 or $P_0^1 \rtimes C_2$.

Now we show, that for every involution $j \in P_0 I$ if we take $p_1 \in C_{P \setminus P_0}(j)$, which is nontrivial, then $H^{(p_1)} = P_0^1 \langle j \rangle$. By Theorem 2.1 b) $p_1 \in P' \setminus Z(P)$ and $p_1 \in P' \setminus P_0^1$. We show that $p_1 \in P' \setminus P_0^1 Z(P)$. For this consider the representation of the Sylow 3-subgroup of G in Theorem 2.1 h). By Theorem 2.1 b) $C_P(j) \cap Z(P) = \{1\}$. We prove that for every $b \in GF(q)$ there is at most one $c \in GF(q)$ such that $(0, b, c) \in C_P(j)$. Otherwise, if $(0, b, c_1), (0, b, c_2) \in C_P(j)$, then their quotient, $(0, 0, c_1 - c_2) \in C_P(j) \cap Z(P) = \{1\}$. Hence $c_1 = c_2$. Since $|C_P(j)| = q$, for every $b \in GF(q)$ there is exactly one $c \in GF(q)$ with $(0, b, c) \in C_P(j)$. Similar statement holds for $C_{P_0}(j)$. Hence if $b \in GF(3)$, then the unique c must be in $GF(3)$ and so $C_{P_0^1 Z(P)}(j) = C_{P_0}(j)$. Accordingly, $p_1 \in (P' \setminus P_0^1) \cap C_P(j)$ implies that $p_1 \in P' \setminus P_0^1 Z(P)$. Since $p_1 \in N_G(P)$, we have seen at the beginning of the proof that $H^{(p_1)} = (P_0 I)^{(p_1)}$. Using that $p_1 \in P' \setminus P_0^1 Z(P)$, we proved already that $(P_0 I)^{(p_1)}$ is isomorphic to P_0^1 or $P_0^1 \langle i \rangle$. By an analogue of Lemma 7.6 for H , we have that $j \in (P_0 I)^{(p_1)}$. Hence $H^{(p_1)} = P_0^1 \langle j \rangle$.

Finally we prove, that there is an element $p_1 \in P' \setminus P_0^1 Z(P)$ such that $(P_0 I)^{(p_1)}$ does not contain any involutions. Let $j \in (P_0 I)^{(p_1)}$ be a fixed involution. Using Lemma 7.6 for $x = p_1 \in P' \setminus P_0^1 Z(P)$, by Theorem 2.1 (b) and (h) we have that

$$\begin{aligned} |\{p_1 \in P' \setminus P_0^1 Z(P) \mid j \in (P_0 I)^{(p_1)}\}| &= |(P' \setminus P_0^1 Z(P)) \cap (P_0 C_P(j))| \leq \\ &\leq |P_0^1 C_P(j) \setminus P_0^1| = 3(q - 3). \end{aligned}$$

If for some $p_1 \in P' \setminus P_0^1 Z(P)$, the subgroup $(P_0 I)^{(p_1)}$ contains an involution j , then $(P_0 I)^{(p_1)} = P_0^1 \langle j \rangle$. Since the involutions are conjugate in $P_0 I$ by elements of P_0 , for every involution $j \in P_0 I$ the same number of $p_1 \in P' \setminus P_0^1 Z(P)$ occurs such that $(P_0 I)^{(p_1)}$ contains j . Since by Lemma 2.4 we have that $C_P(p_1) = P'$, the cosets of P_0^1 in P_0 move p_1 to 3 different places. Thus, from the number of elements p_1 belonging to the involution j , one can get an upper bound on the number of elements p_1 belonging to any involution in $P_0 I$, by multiplying with 3. Since $P_0 I$ contains 3^2 involutions, there are 3 subgroups. Thus we have that

$$|\{p_1 \in P' \setminus P_0^1 Z(P) \mid (P_0 I)^{(p_1)} \text{ contains involutions}\}| \leq 9(q - 3).$$

Since $|P' \setminus P_0^1 Z(P)| = q(q - 3)$ is bigger than $9(q - 3)$, there is an element $p_1 \in P' \setminus P_0^1 Z(P)$ such that $(P_0 I)^{(p_1)}$ does not contain involutions. Then, by the beginning of the proof of II., we have that $(P_0 I)^{(p_1)} = P_0^1$ and hence $H^{(p_1)} = (P_0 I)^{(p_1)} = P_0^1$.

III. If $r^{-1}g = p_1 \in P \setminus P_0P'$ then $H^{(g)} = (P_0I)^{(p_1)} \geq Z(P_0)$. We show that $(P_0I)^{(p_1)}$ cannot contain noncentral elements of order 3 in P_0 . As before, we use the representation in Theorem 2.1 (h) of P and P_0 . Let us suppose that $h \in (P_0I)^{(p_1)}$ is a noncentral element of order 3 in P_0 . Let $p_1^{-1} = (a, b, c)$ and $h = (0, y_0, z_0) \in P_0$. Then $a \notin GF(3)$, otherwise $(a, b, c) = (a, 0, 0)(0, b, c + ab) \in P_0P'$, which is not the case. On the other hand, $y_0 \neq 0$, since otherwise $h \in Z(P_0)$. Then by Lemma 2.6, $h^{p_1^{-1}} = (0, y_0, z_0 + 2y_0a) \in P_0$, which is a contradiction.

Now we prove that $H^{(g)} = (P_0I)^{(p_1)}$ does not contain any elements outside P_0 .

Suppose this is not true and let j be an involution in $(P_0I)^{(p_1)}$. By Lemma 7.6 we have that $p_1 \in P_0C_P(j)$ and so by Theorem 2.1 (b) we have that $p_1 \in P_0P'$, which contradicts our assumption.

Now we show that if there is an element $h \in P_0$ of order 9 in $(P_0I)^{(p_1)}$ then $(P_0I)^{(p_1)} = Z(P_0)\langle h \rangle$. It also follows as in the proof of Lemma 7.8, but in this special case, it also trivially follows from the fact that $Z(P_0)\langle h \rangle = \langle h \rangle \simeq C_9$ is a maximal subgroup in P_0 .

Thus we have seen that $p_1 \in P \setminus P_0P'$ implies that $(P_0I)^{(p_1)}$ can be either $Z(P_0)\langle h \rangle$ for some element $h \in P_0$ of order 9, or $(P_0I)^{(p_1)} = Z(P_0)$.

We show that there is an element $p_1 \in P \setminus P_0P'$ such that $H^{(p_1)} = (P_0I)^{(p_1)} = Z(P_0)$. Let $p_1^{-1} = (a, b, c) \in P \setminus P_0P'$. Let $h = (x_0, y_0, z_0) \in P_0$ be an element such that $x_0 \neq 0$, i.e. it is of order 9, and let $h \in (P_0I)^{(p_1)}$. Then $h^{p_1^{-1}} \in P_0$ hence x_0, y_0, z_0 satisfies

$$x_0(a\sigma) - a(x_0\sigma) \in GF(3) \quad (\text{I})$$

and

$$-2x_0b + 2y_0a - ax_0(x_0\sigma) + ax_0(a\sigma) \in GF(3) \quad (\text{II})$$

Let $(x_0, y_0, z_0) \in P_0$ be fixed such that $x_0 \neq 0$. Then

$$\begin{aligned} & |\{p_1 \in P \setminus P_0P' \mid (x_0, y_0, z_0) \in (P_0I)^{(p_1)}\}| \\ & \leq |\{(a, b, c) \in GF(q)^3 \mid a \notin GF(3), \text{ and } b \text{ satisfies (II)}\}| \leq (q-3)3q. \end{aligned}$$

If $p_1 \in P \setminus P_0P'$ and $(P_0I)^{(p_1)}$ contains an element h of order 9, then $(P_0I)^{(p_1)} = Z(P_0)\langle h \rangle$, thus it contains exactly 6 elements of order 9.

So there are $\frac{3^3-3^2}{6} = 3$ elements of order 9 in P_0 giving different subgroups in P_0 isomorphic to $Z(P_0)\langle h \rangle$. Thus

$$|\{p_1 \in P \setminus P_0P' \mid (P_0I)^{(p_1)} \text{ contains elements of order 9}\}| \leq 9q(q-3).$$

Since $|P \setminus P_0P'| = q^2(q-3)$ is bigger than $9q(q-3)$, we have that there is an element p_1 such that $H^{(p_1)} = (P_0I)^{(p_1)} = Z(P_0)$.

(B) Since $H^{(g)}$ does not contain 3-elements and $H^{(g)} \leq N_H(P_0) = P_0I$, we have that $H^{(g)}$ is isomorphic to 1 or C_2 . \square

Corollary 8.3. *Using notation of Proposition 8.1 we have that*

$$U_{N_H(P_0)} = \{Z(P_0), P_0^1, P_0^1\langle i^{p_0} \rangle \mid p_0 \in P_0\} \cup U \cup V,$$

where $V \subseteq \{Z(P_0)\langle h \rangle, P_0, P_0I \mid h \in P_0 \setminus P_0^1, p_0 \in P_0\}$ and $U \subseteq \{1, \langle i^{p_0} \rangle \mid p_0 \in P_0\}$.

Proposition 8.4. *Using Notation 7.9 and of Prop. 8.1, we have that if $H^{(g)}$ contains a nontrivial element from M_0^+ , then $g \in HN_G(M_0^{+1})$. In particular*

$$U_{N_H(M_0^{+1})} = \{M_0^{+1}, M_0^{+1} \rtimes C_2\} \cup U, \text{ if } 7 \mid q+1 \text{ and}$$

$$U_{N_H(M_0^{+1})} = \{M_0^{+1}\} \cup U, \text{ otherwise,}$$

where U may contain only cyclic subgroups of order 2 or 1. In particular $H^{(g)} = M_0^{+1} \rtimes C_2$ holds for some $g \in G$ iff $7 \mid q+1$ and $g \in H(V \setminus \{1\})$, where V is the Klein subgroup in $N_G(M_0^{+1})$.

Proof. Assume that $g \in G$ such that $H^{(g)}$ contains a nontrivial element $m \in M_0^{+1}$. Then $M_0^{+1} \leq H^{(g)}$, thus $(M_0^{+1})^{g^{-1}} \leq H$. Using Sylow's theorem for $M_0^+ \in \text{Syl}_7(H)$, we have that there is an element $r \in H$ such that $(M_0^{+1})^{g^{-1}r} = M_0^{+1}$ and thus $r^{-1}g \in N_G(M_0^{+1})$. Hence $g \in HN_G(M_0^{+1})$.

To prove the second part, suppose that the element $g \in G$ has the property that $H^{(g)} \leq N_H(M_0^{+1})$. We have seen in Lemma 8.2, that if $H^{(g)}$ contains a nontrivial 3-element, then it contains the center

of a Sylow 3-subgroup of H . However $N_H(M_0^{+1})$ contains only such 3-elements, which centralize an involution. Thus by Proposition 8.1 b) we get a contradiction. Hence, in this case $H^{(g)}$ cannot contain nontrivial 3-elements. If $H^{(g)}$ does not contain nontrivial elements, whose order divides 7, this subgroup can be isomorphic either to C_2 or to $\{1\}$. Assume that $H^{(g)}$ contains nontrivial elements from M_0^{+1} . By the first part of the proof, we have that there is an element $r \in H$ such that $r^{-1}g \in N_G(M_0^{+1})$. Using Lemma 7.1 b) with $T_0 := H$, $H := N_G(M_0^{+1})$ and $t := r$, we have that

$$H^{(g)} = (H \cap N_G(M_0^{+1}))^{(r^{-1}g)} = (N_H(M_0^{+1}))^{(r^{-1}g)},$$

where $r^{-1}g \in N_G(M_0^{+1}) \setminus H$. Then $M_0^{+1} \leq N_H(M_0^{+1})^{(r^{-1}g)}$ by the choice of $r^{-1}g$. Recall that $N_H(M_0^{+1}) = M_0^{+1} \rtimes \langle t_1 \rangle$, where t_1 is an element of order 6. Observe that both $N_H(M_0^{+1})$ and $N_H(M_0^{+1})^{r^{-1}g}$ are Frobenius groups with the same Frobenius kernel. Thus $N_H(M_0^{+1})^{(r^{-1}g)} \neq M_0^{+1}$, iff there is an element $r_1 \in M_0^{+1}$ such that

$$\langle t_1 \rangle^{r^{-1}gr_1} \cap \langle t_1 \rangle \neq 1, \quad (*)$$

where $r^{-1}gr_1 \in N_G(M_0^{+1}) \setminus N_H(M_0^{+1})$. Depending on the relation of 3 and q the subgroup M_0^{+1} is contained in one of $\tilde{M}^{+1} \in \text{Hall}_{q+3m+1}(G)$, $\tilde{M}^{-1} \in \text{Hall}_{q-3m+1}(G)$ or $\tilde{M} \in \text{Hall}_{\frac{q+1}{4}}(G)$.

First assume that $M_0^{+1} \leq \tilde{M}^{+1}$. Then $N_G(M_0^{+1}) = \tilde{M}^{+1} \rtimes \langle t_1 \rangle$ is also a Frobenius group with Frobenius complement $\langle t_1 \rangle$. The equation (*) implies that $r^{-1}gr_1 \in \langle t_1 \rangle$, which is a contradiction. Thus $H^{(g)} = M_0^{+1}$ in this case. This can really occur, e.g. if we choose $g \in \tilde{M}^{+1} \setminus H$.

If $M_0^{+1} \leq \tilde{M}^{-1}$, then the proof is similar.

Finally let us assume that $M_0^{+1} \leq \tilde{M}$. Thus $N_G(M_0^{+1}) = (\tilde{M} \times V) \rtimes \langle t_1 \rangle$. Let m be a generator of \tilde{M} and $r^{-1}gr_1 = t_1^a m^b v_1$, where $a, b \in \mathbb{Z}$, and $v_1 \in V \setminus \{1\}$. Suppose that the 3-element t_1^2 acts on $V = \{1, v_1, v_2, v_3\}$ as $v_1^{t_1^2} = v_2$, $v_2^{t_1^2} = v_3$ and $v_3^{t_1^2} = v_1$. The involution t_1^3 centralizes V , thus $v_1^{t_1^3} = v_3$ etc. We have that $t_1^{a m^b v_1} = t_1^{m^b v_1} = (t_1[t_1, m^b])^{v_1} = t_1^{v_1}[t_1, m^b] = t_1[t_1, v_1][t_1, m^b]$. By Lemma 2.3, $[t_1, m^b] \neq 1$ if and only if $\frac{q+1}{4} \nmid b$. Thus $\langle t_1 \rangle^{t_1^{a m^b v_1}} \cap \langle t_1 \rangle = \{1\}$, if $\frac{q+1}{4} \nmid b$. However, $\langle t_1 \rangle^{t_1^{a v_1}} = \langle t_1 \rangle^{v_1}$ and since $t_1^{v_1} = v_3 t_1$, $(t_1^{v_1})^2 = v_2 t_1^2$, $(t_1^{v_1})^3 = t_1^3$, we have that $\langle t_1 \rangle^{t_1^{a v_1}} \cap \langle t_1 \rangle = \langle t_1^3 \rangle$. Thus if $H^{(g)}$ contains nontrivial elements from M_0^{+1} and $M_0^{+1} \leq \tilde{M}$ then $H^{(g)}$ can be either M_0^{+1} or $M_0^{+1} \rtimes \langle t_1^3 \rangle$. These cases in fact occur. If $g \in \tilde{M} \setminus M_0^{+1}$, then $H^{(g)} = M_0^{+1}$ and if $g \in V \setminus \{1\}$, then $H^{(g)} = M_0^{+1} \rtimes \langle t_1^3 \rangle$. □

Proposition 8.5. *If $H^{(g)} \neq H$, then it is contained in $N_H(P_0)$ or $N_H(M_0^{+1})$ for some $P_0 \in \text{Syl}_3(H)$, $M_0^{+1} \in \text{Syl}_7(H)$ or it is isomorphic a Klein four group.*

Proof. Let assume that $S \in \text{Syl}_2(H)$ is in $H^{(g)}$ for some $g \in G$. Since $S, S^{g^{-1}}$ are Sylow 2-subgroups of H , there exists an element $r \in H$ such that $g^{-1}r \in N_G(S)$. However, $N_G(S) \leq H$ and so $g \in H$. Thus $H^{(g)} = H$.

Let assume that $M_0^{+1} \simeq C_7$ is contained in $H^{(g)} \neq H$ for some $g \in G$. Thus $H^{(g)}$ is contained in a subgroup of H , maximal with the property that it has some 7-elements but not having C_2^3 as a subgroup. The maximal such subgroup in $N_H(S)$ or in H' is isomorphic to $C_7 \rtimes C_3$ or to D_{14} , respectively. Since both of them are contained in a conjugate of $N_H(M_0^{+1})$, we can suppose that $M_0^{+1} \leq H^{(g)} \leq N_H(M_0^{+1})$.

The largest subgroup of H' containing 3-elements is isomorphic to D_{18} . This is a subgroup of $N_H(P_0)$ for suitable $P_0 \in \text{Syl}_3(H)$. If $H^{(g)} \neq H$ and it is not contained up to conjugacy in any of the subgroups $N_H(P_0)$, $N_H(M_0^{+1})$ and H' , then $H^{(g)}$ is a subgroup of $N_H(S)$ for some $S \in \text{Syl}_2(H)$, not containing S and 7-elements.

Thus, it is isomorphic to A_4 or a Klein four group. The subgroups of size 12 are all conjugate in H . However, since for an involution $i \in H$, the subgroup $C_H(i) \simeq \langle i \rangle \times A_4$, hence by Lemma 8.1, the involution of A_4 cannot be in the center of a Sylow 3-subgroup of H . So, by Lemma 8.2, A_4 cannot occur, either. □

Proposition 8.6. *Let H be a maximal subgroup of G isomorphic to $R(3)$. Then $\mathcal{U}_1(H) = \mathcal{U}_2(H)$. In particular $d_c(H, G) = 4$ and $d(H, G) = 3$.*

Proof. From the previous proposition we get that $H^{(g)}$ can be isomorphic to one of the following: H , C_7 or D_{14} , $Z(P_0) \simeq C_3$, $P_0^1 \simeq C_3^2$, $P_0^1 \rtimes C_2 \simeq C_3^2 \rtimes C_2$, C_9 (two classes), P_0 , $P_0 \rtimes C_2$, C_2 , 1, and C_2^2 . Since $H^{(g_1, g_2)} = H^{(g_1)} \cap H^{(g_2)}$, to finish the proof it is enough to show that \mathcal{U}_1 is

intersection closed. However, we already have seen in Proposition 8.5 that there is an $g \in G$ such that $H^{(g)}$ is isomorphic to $Z(P_0)$, P_0^1 , $P_0^1 \rtimes C_2$. The Klein subgroup could only occur in $\mathcal{U}_2(H)$ as the intersection with itself, it does not occur as the intersection of other subgroups in $\mathcal{U}_1(H)$. If we show, that $1, C_2$ are in $\mathcal{U}_1(G)$, then we see that $\mathcal{U}_1(G) = \mathcal{U}_2(G)$. We have to show that there are elements $g_1, g_2 \in G$ such that $H^{(g_1)} = 1$ and $H^{(g_2)} \simeq C_2$. Now we give constructions for these cases.

Let i be an involution and $C_G(i) = \langle i \rangle \times L$, where $L \simeq PSL_2(q)$ and let $C_H(i) = \langle i \rangle \times A$, where $A \simeq A_4$. By [3, Lemma 2.7] there is an element $l \in L$ such that $A^{(l)} = 1$. Obviously $H^{(l)}$ contains i and if $H^{(l)} \neq \langle i \rangle$, then $H^{(l)} \simeq M_0^{+1} \rtimes \langle i \rangle$ in the case $7 \mid q+1$, since $H^{(l)} \cap C_G(i) = \langle i \rangle$. However, by Proposition 8.4 we know that $H^{(l)}$ can be $M_0^{+1} \rtimes \langle i \rangle$, if and only if $l \in H(V \setminus \{1\})$ and $7 \mid q+1$, where V is the Klein subgroup in $N_G(M_0^{+1})$. Since $l \in L$ and we can suppose that $V \leq L$, we get that

$$|\{l \in L \mid H^{(l)} = M_0^{+1} \rtimes \langle i \rangle\}| = |H(V \setminus \{1\}) \cap L| = |A|3 = 36.$$

However, every involution is in 4 conjugates of $N_H(M_0^{+1})$ and so

$$|\{l \in L \mid H^{(l)} \simeq C_7 \rtimes \langle i \rangle \text{ and contains } i\}| = 36 \cdot 4 = 144.$$

We will show there are more than 144 elements in L such that $A^{(l)} = 1$.

According to the proof of [3, Lemma 2.7] we have that

$$|\{A^l \mid l \in L \text{ and } A^{(l)} = 1\}| \geq |L : N_L(A)| - |\{A^l \neq A \mid l \in L \text{ and } A^{(l)} \text{ contains involutions}\}| - |\{A^l \neq A \mid l \in L \text{ and } A^{(l)} \text{ contains order 3 elements}\}| - 1 \geq$$

$$\frac{q^3 - q}{24} - 3\left(\frac{q+1}{4} - 1\right) - 4\left(\frac{q}{3} - 1\right) - 1 = \frac{q^3 - q}{24} - \frac{25q - 63}{12},$$

which is bigger than 144 if $q \geq 27$. Hence $|\{l \in L \mid A^{(l)} = 1\}| > 144$.

Thus there is an element $l \in L$ such that $A^{(l)} = 1$ and $H^{(l)} = \langle i \rangle$.

If $H^{(g)}$ does not contain any 2-elements, 3-elements or 7-elements, then $H^{(g)} = 1$. Let i be an involution in H . Using the fact that $i \in H^{(g)}$ for some $g \in G$ iff $g \in HC_G(i)$ from Lemma 8.2 and Proposition 8.4 we get that

$$\begin{aligned} & \bigcup_{P_0 \in \text{Syl}_3(H)} \{g \in G \mid P_0 \cap H^{(g)} \neq 1\} \cup \bigcup_{i \in H} \bigcup_{o(i)=2} \{g \in G \mid i \in H^{(g)}\} \cup \bigcup_{M_0^{+1} \in \text{Syl}_7(H)} \{g \in G \mid M_0^{+1} \leq H^{(g)}\} = \\ & \bigcup_{P_0 \in \text{Syl}_3(H), P_0 \leq P \in \text{Syl}_3(G)} HN_G(P) \cup \bigcup_{i \in H} \bigcup_{o(i)=2} HC_G(i) \cup \bigcup_{M_0^{+1} \in \text{Syl}_7(H)} HN_G(M_0^{+1}). \end{aligned}$$

The size of this is at most

$$28^2 \cdot q^3(q-1) + 63^2 \cdot (q^3 - q) + 36^2 \cdot 6 \cdot (q + 3m + 1).$$

Since this is smaller than $|G|$, there is an element $g \in G$ such that $H^{(g)} = 1$.

Since $\mathcal{U}_1(H) = \mathcal{U}_2(H)$, the depth of H in G is 3 or 4. Using Definition 1.1 (ii) we show that $d_c(H, G) > 3$. Let $x_2 \in G$ be such that $H^{(x_2)} = P_0^1$ and let $x_1 \in G_0 \setminus N_H(P_0')$. Then $H^{(x_1, x_2)} = P_0'$. Assume by contradiction that there exists an element $y_1 \in G$ such that $H^{(x_1, x_2)} = G_0^{(y_1)}$ and $h^{x_1} = h^{y_1}$ for all $h \in P_0^1$. Since $(P_0^1)^{x_1} = (P_0^1)^{y_1}$, we have that $(P_0^1)^{x_1} \leq H^{y_1}$ and so $(P_0^1)^{x_1} \leq H^{(y_1)} = P_0^1$. By the choice of x_1 , i. e. $x_1 \notin N_G(P_0')$, we have a contradiction. Thus $d_c(H, G) > 3$.

Using Theorem 1.3 we get that $d(H, G) = 3$. □

Acknowledgements: The first and the second author were supported by the National Research, Development and Innovation Office -NKFIH Grant No. 115288 and 115799.

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